

Conjugacy invariants for Brouwer mapping classes

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Abstract

We give new tools for homotopy Brouwer theory. In particular, we describe a canonical reducing set (the set of *walls*) which splits the plane into *maximal translation areas* and *irreducible areas*. We then focus on Brouwer mapping classes relatively to four orbits and describe them explicitly by adding to Handel's diagram and to the set of walls a *tangle*, which is essentially an isotopy class of simple closed curves in the cylinder minus two points.

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Introduction

Homotopy Brouwer theory

Homotopy Brouwer theory was introduced by Michael Handel in [Han99] to prove his famous fixed point theorem for planar homeomorphism, which has many applications

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to surface homeomorphisms (for examples, see the introduction of [LC06]). This theory was mainly used by John Franks and Michael Handel, e.g. to study Hamiltonian surface diffeomorphisms in [FH03b] and prove the Zimmer conjecture for area preserving diffeomorphisms of surfaces in [FH03a].

Homotopy Brouwer theory can be seen as the study of the elements of the mapping class group of the plane minus \mathbb{Z} which are classes of Brouwer homeomorphisms relatively to finitely many orbits. More precisely, consider a Brouwer homeomorphism h , i.e. a fixed-point-free homeomorphism of the plane preserving the orientation. Choose finitely many disjoint orbits of this homeomorphism and denote by \mathcal{O} their union. Classical Brouwer theory tells us that each orbit of a Brouwer homeomorphism is properly embedded in the plane, i.e. intersects every compact set of the plane in only finitely many points (see for example [Gui94]). Hence \mathcal{O} is homeomorphic to \mathbb{Z} in the plane. Denote by $MCG(\mathbb{R}^2, \mathcal{O})$ the mapping class group of the plane relatively to \mathcal{O} , i.e. the quotient of the group of orientation preserving homeomorphisms of the plane which globally fix \mathcal{O} by its connected component of the identity for the compact-open topology. Now we can look at the class of h in $MCG(\mathbb{R}^2, \mathcal{O})$: since h is a Brouwer homeomorphism, this class is said to be a *Brouwer mapping class*. We denote it by $[h; \mathcal{O}]$. Two Brouwer mapping classes $[h; \mathcal{O}]$ and $[h'; \mathcal{O}']$ are said to be conjugate if there exists an orientation-preserving homeomorphism ϕ of the plane such that $\phi(\mathcal{O}) = \mathcal{O}'$ and $[\phi h \phi^{-1}; \phi(\mathcal{O})]$ is equal to $[h'; \mathcal{O}']$ in $MCG(\mathbb{R}^2; \mathcal{O}')$. One aim of homotopy Brouwer theory is to describe (up to conjugacy) every Brouwer mapping class relatively to finitely many given orbits.

Brouwer mapping classes relatively to one, two, and three orbits

In [Han99], Michael Handel gives a complete description of Brouwer mapping classes relatively to one and two orbits. He shows that relatively to one orbit, there exists only one Brouwer mapping class up to conjugacy: the class of the translation relatively to one of its orbits. Relatively to two orbits, he proves that they are exactly three Brouwer mapping classes (up to conjugacy): the class of the translation, the class of the time one map R of the Reeb flow and the class of R^{-1} .

In [LR13], Frédéric Le Roux gives a complete description of Brouwer mapping classes relatively to 3 orbits and uses this description to define an index for Brouwer homeomorphisms. In particular, he shows that there are only finitely many Brouwer mapping classes relatively to 3 orbits, and that each of them contains the time one map of a flow (see [LR13] for more details and the complete description of this classes).

The situation changes if we look at the Brouwer mapping classes relatively to more than 3 orbits: indeed, if $r \geq 4$, there are infinitely many Brouwer mapping classes relatively to r orbits, and only finitely many of them contain the time one map of a flow. One aim of this paper is to give a complete description of Brouwer mapping classes relatively to 4 orbits.

Walls

In [Han99], Michael Handel defines reducing lines of a Brouwer mapping class $[h; \mathcal{O}]$: such a line is homotopic to its image by h and splits the set of orbits into two smaller sets. He proves that every Brouwer mapping class relatively to more than one orbits has at least one reducing line (theorem 2.7 of [Han99]).

We propose to call the isotopy class of a reducing line Δ a *wall* if every other reducing line is homotopically disjoint from Δ . The set of walls is clearly a conjugacy invariant for Brouwer mapping classes. We prove the following result:

Theorem. 2.5 *Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Let \mathcal{W} be a family of pairwise disjoint reducing lines containing exactly one representative of each wall for $[h; \mathcal{O}]$. If Z is a connected component of $\mathbb{R}^2 - \mathcal{W}$, then exactly one of the followings holds:*

- *Z is an irreducible area;*
- *Z is a maximal translation area;*
- *Z does not intersect \mathcal{O} .*

Precise definitions of irreducible and maximal translation areas will be given in section 2. A translation area Z is in particular an area which is invariant under h and on which h has a very simple dynamics: indeed, up to conjugacy, h is conjugated to a homeomorphism whose restriction to Z is a translation. Contrariwise, an irreducible area has a more complex dynamics and cannot be reduced into more simple areas: it does not contain any reducing line. It will follow that if the complement of the walls of a Brouwer mapping class does not contain any irreducible area, then we will be able to understand easily the Brouwer mapping class, which will indeed be a time one map of a flow (see section 2). Moreover, we will prove that if the complement of the walls of a Brouwer mapping class has an irreducible area, then it has at least also two maximal translation areas.

Diagrams

Following essentially [Han99] and [LR13], we can associate to every conjugacy class of Brouwer mapping class a unique diagram, for which a precise definition will be given in section 2. This diagram is a disk with r arrows, where r is the number of orbits that we consider: each arrow represents an orbit. The cyclic order of the endpoints of the arrows is determined by the existence of a *nice family of homotopy translation arcs* (see section 2). A diagram of Brouwer mapping class is said to be *determinant* if there exists only one conjugacy class of Brouwer mapping class associated to it. Every diagram for Brouwer mapping class relatively to one, two or three orbits is determinant. For 4 orbits or more, there exist diagrams which are not determinant.

We can add the set of walls on this diagram: we obtain a *diagram with walls*, which is again a conjugacy invariant for Brouwer mapping classes (see figure 1 for examples). This invariant is more precise than the diagram without walls, but it is still not total for

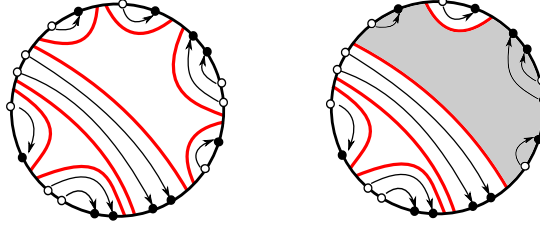


Figure 1: Examples of diagrams with walls: the one on the left is determinant, and the one on the right is non determinant

Brouwer mapping classes relatively to more than 3 orbits. Again we can define the notion of *determinant* diagram with walls (which corresponds to only one conjugacy class). We give an elementary combinatoric condition to identify the determinant diagrams among diagrams with walls without crossing arrows:

Proposition. 2.8 *A diagram with walls without crossing arrows is determinant if and only if the arrows of every family of arrows included in the same connected component of the complement of the walls are backward adjacent and forward adjacent.*

A total conjugacy invariant for Brouwer mapping class relatively to 4 orbits

For Brouwer mapping classes relatively to 4 orbits, we add a new invariant to the non determinant diagrams with walls: the *tangle*. This invariant is an isotopy class of curves on the cylinder with two marked points (up to horizontal twists). See figure 2 for an example. Using in particular the set of walls and the description of determinant diagrams with 4 orbits, we get a total conjugacy invariant:

Theorem. 2.13 *Two Brouwer mapping classes relatively to 4 orbits are conjugated if and only if they have the same couple (Diagram with walls, Tangle).*

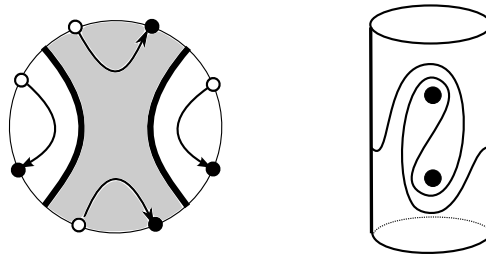


Figure 2: Example of a couple (Diagram with walls, Representative of the tangle)

We finally get a complete description of Brouwer mapping classes relatively to four orbits.

In the first section, we recall useful tools for homotopy Brouwer theory from [Han99] and [LR13]. A precise description of the results will be given in section 2. The remaining of the text is devoted to proofs.

Acknowledgment

I would like to thank my advisor Frédéric Le Roux for his many advices and explanations and for his careful readings of the different versions of this text.

1 First tools of homotopy Brouwer theory

We recall the following definitions and properties (see [Han99], [LR12] and [LR13]). Let $[h; \mathcal{O}]$ be a Brouwer mapping class, i.e. the isotopy class of a Brouwer homeomorphism h relatively to a finite set of orbits \mathcal{O} . Denote by r the number of orbits of \mathcal{O} . We choose a complete hyperbolic metric of the first kind on $\mathbb{R}^2 - \mathcal{O}$. Even if not explicitly specified, we will always consider complete hyperbolic metrics *of the first kind* on surfaces, i.e. such that the surface is isomorphic to \mathbb{H}^2/Γ where Γ is of the first kind (see Matsumoto [Mat00] for details).

1.1 Examples: flows and product with a free half twist

Flows. For abbreviation, we say that a homeomorphism f is a flow if it is the time one map of a flow. If a Brouwer homeomorphism f is isotopic to a flow relatively to \mathcal{O} , then we say that $[f; \mathcal{O}]$ is a *flow class*.

Example A. The first example is the flow class of figure 3. In this example, we choose 5 streamlines of a flow f and get a Brouwer mapping class relative to 5 orbits: $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$ and \mathcal{O}_5 . We denote by \mathcal{O} their union.

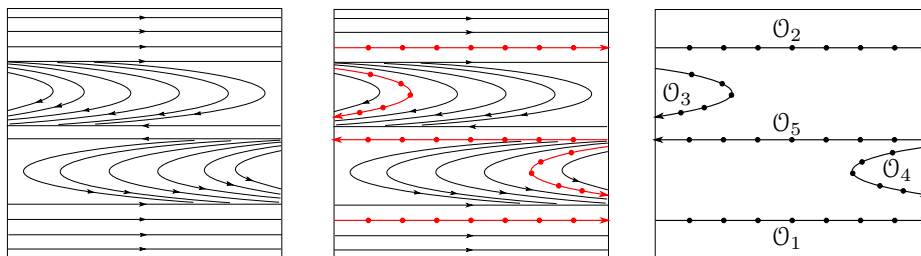


Figure 3: Example of a Brouwer mapping class $[f; \mathcal{O}]$ relatively to 5 orbits.

Free half twist. We call *half twist* any homeomorphism which is:

- supported in a topological disk D of \mathbb{R}^2 which contains exactly 2 points of \mathcal{O} , denoted by x and y ;

- isotopic to a homeomorphism supported in D which is a rotation of a half turn on a disk included in D , which exchanges x and y .

If h is a Brouwer homeomorphism, we call *h -free half twist* every half twist μ supported in a h -free disk, i.e. in a disk D such that $h^n(D) \cap D = \emptyset$ for every non zero $n \in \mathbb{Z}$. Note that μh is a Brouwer homeomorphism.

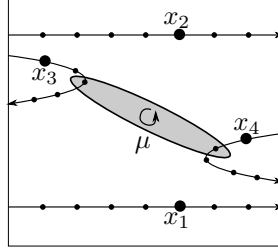


Figure 4: Example B : the Brouwer mapping class $[g; \mathcal{O}']$.

Example B. Our second example is the product of f with the free half twist μ of figure 4, which exchanges the two points of the disk which are in \mathcal{O} . We denote by g the product μf . For $i = 1, 2, 3, 4$, we choose x_i on \mathcal{O}_i as in figure 4 and denote by \mathcal{O}'_i the g -orbit of x_i , i.e. $\{g^n(x_i)\}_{n \in \mathbb{Z}}$. In particular, we have $\mathcal{O}_1 = \mathcal{O}'_1$ and $\mathcal{O}_2 = \mathcal{O}'_2$. We denote by \mathcal{O}' the union of the \mathcal{O}'_i . Note that \mathcal{O}' coincide with $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$. We consider the Brouwer mapping class $[g; \mathcal{O}']$.

1.2 Arcs and topological lines in $\mathbb{R}^2 - \mathcal{O}$.

We call *arc* an embedding α of $]0, 1[$ in $\mathbb{R}^2 - \mathcal{O}$ such that it can be continuously extending to 0 and 1 with $\alpha(0), \alpha(1) \in \mathcal{O}$. By abuse of notations, we will call again arc and denote by α the image of the embedding $\alpha(]0, 1[)$. The extensions $\alpha(0)$ and $\alpha(1)$ are said to be the *endpoints* of α . A *topological line* is a proper embedding α of \mathbb{R} in \mathbb{R}^2 , i.e. an embedding α such that for every compact K of the plane, there exists $t_0 \in \mathbb{R}$ such that if $|t| > |t_0|$, then $\alpha(t) \notin K$. Again, by abuse of notations, we call (topological) line and denote by α the image of \mathbb{R} by α in $\mathbb{R}^2 - \mathcal{O}$.

1.3 Isotopy classes of arcs and lines.

We say that two arcs (respectively two lines) α and β are isotopic relatively to \mathcal{O} if there exists a continuous and proper application $H :]0, 1[\times [0, 1] \rightarrow \mathbb{R}^2 - \mathcal{O}$ such that:

- $H(\cdot, 0) = \alpha(\cdot)$ and $H(\cdot, 1) = \beta(\cdot)$;
- If α and β are arcs, H can be continuously extending to $[0, 1] \times [0, 1]$ in such a way that the endpoints coincide, i.e. for every $t \in]0, 1[$, we have: $\alpha(0) = H(0, t) = \beta(0)$ and $\alpha(1) = H(1, t) = \beta(1)$.

If α is an arc or a line of $\mathbb{R}^2 - \mathcal{O}$, we denote by $\alpha_{\#}$ the geodesic representative in the isotopy class of α relatively to \mathcal{O} . It is known that this geodesic representative is unique and that if α and β are two arcs or lines, then $\alpha_{\#}$ and $\beta_{\#}$ are in minimal position. In particular, if α and β are homotopically disjoint (i.e. they have disjoint representatives in their isotopy classes), then $\alpha_{\#}$ and $\beta_{\#}$ are disjoint.

1.4 Straightening principle.

We will need the following lemma, which is the lemma 3.5 of [Han99] (see also lemma 1.4, corollary 1.5 and lemma 3.2 of [LR13]):

Lemma 1.1 (Straightening principle). *Let \mathcal{F}_1 and \mathcal{F}_2 be two locally finite families of lines and arcs of $\mathbb{R}^2 - \mathcal{O}$ such that:*

- *The elements of \mathcal{F}_1 (respectively \mathcal{F}_2) are mutually homotopically disjoint;*
- *If $\alpha \in \mathcal{F}_1$ and $\beta \in \mathcal{F}_2$, then α and β are in minimal position.*

Then the following statements hold.

1. *There exists a homeomorphism u isotopic to Id relatively to \mathcal{O} such that for every element γ in $\mathcal{F}_1 \cup \mathcal{F}_2$, $u(\gamma) = \gamma_{\#}$, where $\gamma_{\#}$ is the geodesic representative of the isotopy class of γ ;*
2. *If h is an orientation preserving homeomorphism of the plane such that $h(\mathcal{O}) = \mathcal{O}$, then there exists $h' \in [h; \mathcal{O}]$ such that for every α in $\mathcal{F}_1 \cup \mathcal{F}_2$, we have $h'(\alpha_{\#}) = h(\alpha)_{\#}$.*

We get 2 by applying 1 to $h(\mathcal{F}_1 \cup \mathcal{F}_2)$.

1.5 Homotopy translation arc.

A *homotopy translation arc* for $[h; \mathcal{O}]$ is an arc α such that:

- There exists $x \in \mathcal{O}$ such that $\alpha(0) = x$ and $\alpha(1) = h(x)$;
- For every $n \in \mathbb{Z}$, $h^n(\alpha)$ is homotopically disjoint from α .

In particular, every translation arc for h with endpoints in \mathcal{O} is a homotopy translation arc for $[h; \mathcal{O}]$. In general, there exist homeomorphisms h and arcs which are not homotopic to translation arc for h , but which are homotopy translation arcs for $[h; \mathcal{O}]$.

Example A: Figure 5 (left) shows different homotopy translation arcs for example A: the arcs β and γ are homotopy translation arcs for $[f; \mathcal{O}']$. The arc α is not a homotopy translation arc for $[f; \mathcal{O}]$. Note that however if we forget the orbit \mathcal{O}_5 , α is a homotopy translation arc for $[h; \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4]$.

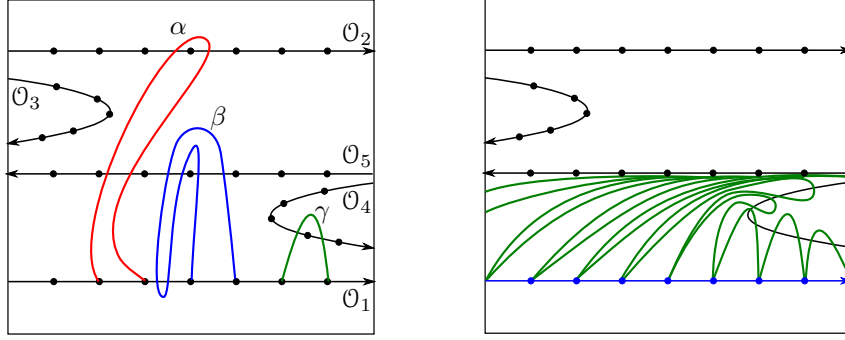


Figure 5: Examples of homotopy translation arcs and homotopy streamlines.

1.6 Half homotopy streamlines.

If α is a homotopy translation arc for $[h; \mathcal{O}]$, we define the *homotopy streamline*:

$$T(\alpha, h, \mathcal{O}) := \bigcup_{n \in \mathbb{Z}} (h^n(\alpha([0, 1]))_{\#}.$$

Since α is a homotopy translation arc, the geodesic iterates are mutually disjoint, hence $T(\alpha, h, \mathcal{O})$ is an embedding of \mathbb{R} , which can eventually be non proper.

We also define the *backward (respectively forward) homotopy streamline* $T^-(\alpha, h, \mathcal{O})$ (respectively $T^+(\alpha, h, \mathcal{O})$) by:

$$T^-(\alpha, h, \mathcal{O}) := \bigcup_{n \leq 0} (h^n(\alpha([0, 1]))_{\#}.$$

$$T^+(\alpha, h, \mathcal{O}) := \bigcup_{n \geq 0} (h^n(\alpha([0, 1]))_{\#}.$$

Example A: The streamline $T(\beta, f, \mathcal{O})$ of example A is proper and coincides with the horizontal streamline which contains \mathcal{O}_1 . The streamline $T(\gamma, h, \mathcal{O})$ is drawn on figure 5. It is not proper, but $T^+(\gamma, h, \mathcal{O})$ is proper.

1.7 Backward proper and forward proper arcs.

Let α be a homotopy translation arc. If the backward homotopy streamline $T^-(\alpha, h, \mathcal{O})$ is the image of \mathbb{R}^+ under a proper embedding, i.e. if for every compact K of the plane there exists $n_0 \leq 0$ such that for every $n \leq n_0$, $(h^n(\alpha))_{\#}$ does not intersect K , then α is said to be a *backward proper arc* for $[h; \mathcal{O}]$. Similarly, if the forward homotopy streamline $T^+(\alpha, h, \mathcal{O})$ is proper, then α is said to be a *forward proper arc* for $[h; \mathcal{O}]$.

Example A: In the example of figure 5, β is a backward proper and forward proper arc, and γ is a forward proper arc but it is not a backward proper arc. Note that if h is a flow, then every homotopy translation arc which lies on a flow streamline is backward

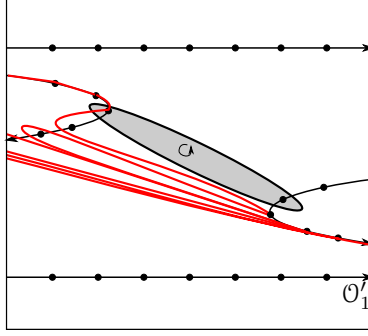


Figure 6: Example of a homotopy streamline for $[g; \mathcal{O}']$.

proper and forward proper.

Example B: In figure 6, we draw iterates of an arc lying on a streamline of f which intersects the support of the free half twist μ . We see that this arc is backward proper but not forward proper (the iterates are “stuck” by the orbit \mathcal{O}'_1).

Translation. If a Brouwer homeomorphism f is conjugate to a translation relatively to \mathcal{O} , then we say that $[f; \mathcal{O}]$ is a *translation class*. In this case, every homotopy translation arc for $[f; \mathcal{O}]$ is backward proper and forward proper.

1.8 Nice family $(\alpha_i^\pm)_{1 \leq i \leq r}$.

A *nice family* $(\alpha_i^\pm)_{1 \leq i \leq r}$ associated to $[h; \mathcal{O}]$ is a family of homotopy translation arcs for $[h; \mathcal{O}]$ such that:

- For every $1 \leq i \leq r$:
 - α_i^- is a backward proper arc;
 - α_i^+ is a forward proper arc;
 - α_i^- and α_i^+ have the same endpoints, lying in the orbit \mathcal{O}_i ;
- The backward proper half streamlines $T^-(\alpha_i^-, h, \mathcal{O})$'s are mutually disjoint;
- The forward proper half streamlines $T^+(\alpha_i^+, h, \mathcal{O})$'s are mutually disjoint.

Note that if $(\alpha_i^\pm)_i$ is a nice family for a Brouwer mapping class $[h; \mathcal{O}]$, then the previous proper half streamlines $T^\pm(\alpha_i^\pm, h, \mathcal{O})$'s are mutually disjoint outside a topological disk of the plane.

Examples: Figure 7 give an example of a nice family for $[f; \mathcal{O}]$ with some arcs not homotopic to arcs included in streamlines, and an example of a nice family for $[g; \mathcal{O}']$. In particular, α_3^+ (respectively α_4^-) is constructed with an iteration by g^{-1} (respectively

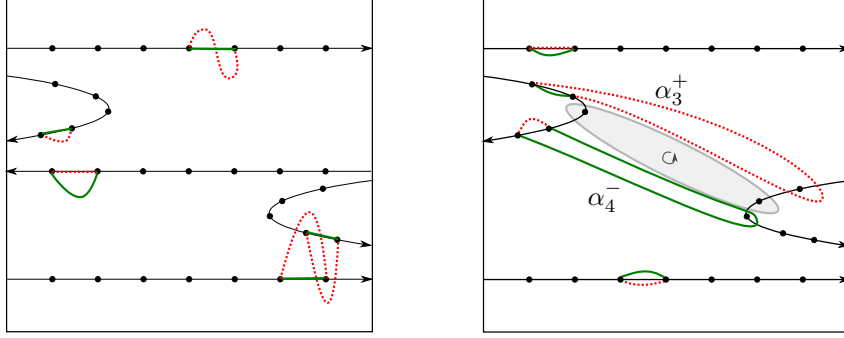


Figure 7: Example of nice families for $[f; \mathcal{O}]$ and $[g; \mathcal{O}']$.

g) of an arc lying on the f -streamline of \mathcal{O}_4 after the support of μ (respectively of an arc lying on the f -streamline of \mathcal{O}_3 before the support of μ).

The following theorem of Handel [Han99] allows us to consider a nice family for every Brouwer mapping class in the following sections.

Theorem 1.2 (Handel [Han99]). *For every $[h; \mathcal{O}]$, there exists a nice family associated to $[h; \mathcal{O}]$.*

Remark 1.1. For a statement closer to this one, see [LR13], proposition 3.1. Here we describe a way to deduce our statement from proposition 3.1 of [LR13], where the statement is given for *generalized homotopy half streamlines*. As seen in figure 8, in any open neighborhood of a generalized homotopy half streamline there exist disjoint homotopy streamlines whose union contains every points of \mathcal{O} included in the generalized homotopy half streamline. It follows that the result is still true with the statement given here (i.e. when we replace disjoint generalized homotopy half streamlines by (non generalized) disjoint homotopy half streamlines).

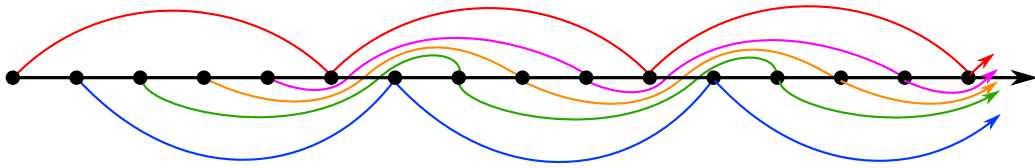


Figure 8: Finding disjoint homotopy half streamlines in a neighborhood of a generalized homotopy half streamline.

1.9 Reducing line.

We say that a line of \mathbb{R}^2 *splits* a given set of points X included in $\mathbb{R}^2 - \Delta$ if both connected components of $\mathbb{R}^2 - \Delta$ intersect X .

A *reducing line* Δ for $[h; \mathcal{O}]$ is a line in $\mathbb{R}^2 - \mathcal{O}$ such that $h(\Delta)$ is properly isotopic to Δ relatively to \mathcal{O} and such that Δ splits \mathcal{O} . Note that all the elements of a same orbit of \mathcal{O} are included in the same connected component of $\mathbb{R}^2 - \Delta$. Indeed, according to the straightening principle 1.1, there exists $h' \in [h; \mathcal{O}]$ such that $h'(\Delta) = \Delta$. Figure 9 gives examples.

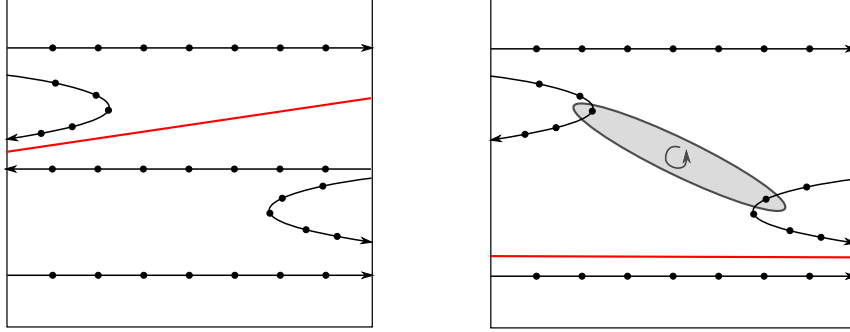


Figure 9: Examples of reducing lines for $[f; \mathcal{O}]$ and $[g; \mathcal{O}']$.

We will need the following Handel's theorem. See [LR13], proposition 3.3 for this formulation.

Theorem 1.3 (Handel [Han99]). *Let $[h; \mathcal{O}]$ be a Brouwer mapping class relatively to more than one orbits. Let $(\alpha_i^\pm)_i$ be a nice family for $[h; \mathcal{O}]$. There exists a reducing line for $[h; \mathcal{O}]$ which is disjoint from every backward proper half streamline $T^-(\alpha_i^-, h, \mathcal{O})$.*

1.10 Homotopy Brouwer line.

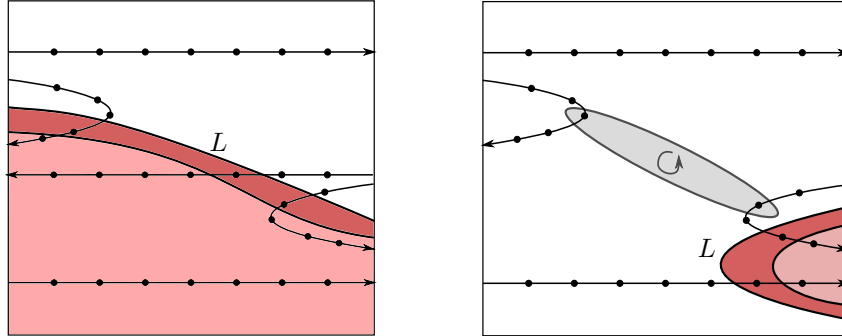


Figure 10: Examples of Brouwer lines for $[f; \mathcal{O}]$ and $[g; \mathcal{O}']$.

A *homotopy Brouwer line* L for $[h; \mathcal{O}]$ is a topological line in $\mathbb{R}^2 - \mathcal{O}$ such that:

- L is homotopically disjoint from $h(L)$;
- L is not isotopic to $h(L)$;

- If we denote by V the connected components of $\mathbb{R}^2 - L_{\#}$ containing $h(L)_{\#}$, then we have $h(V)_{\#} \subset V$, where $h(V)_{\#}$ is the connected component of $\mathbb{R}^2 - h(L)_{\#}$ which does not contain $L_{\#}$;
- If $x \in \mathcal{O}$ is in $V - h(V)_{\#}$, then $h(x) \notin V - h(V)_{\#}$.

This definition does not depend on the chosen metric on $\mathbb{R}^2 - \mathcal{O}$. Figure 10 gives examples of Brouwer lines for $[f; \mathcal{O}]$ and $[g; \mathcal{O}']$.

2 Description of the results

Here we give the main definitions and statements of the paper. Proofs will be given in the following sections.

2.1 Adjacency areas, diagrams and special nice families (Section 3)

2.1.1 Cyclic order of a nice family

Let $[h; \mathcal{O}]$ be a Brouwer mapping class and let $(\alpha_i^{\pm})_i$ be a nice family for $[h; \mathcal{O}]$. There is a natural cyclic order on the elements of the nice family $(\alpha_i^{\pm})_i$ given by the order of the half homotopy streamlines generated by the α_i^{\pm} at infinity: if we choose a big enough topological circle which intersects each half streamline only once, with transverse intersections, the order on the half streamlines is given by the order of these intersections (which is independent of the choice of the circle). In the following, we will call this cyclic order the *cyclic order of the nice family*.

2.1.2 Adjacency

If several forward proper arcs, respectively several backward proper arcs, are consecutive for the cyclic order of the nice family, then they are said to be *adjacent*. A sub-family of the nice family consisting only of consecutive arcs of the same type (all backwards or all forwards) is said to be a *sub-family of adjacency*. If two orbits have forward proper arcs (respectively backward proper arcs) in the same nice family which are adjacent, they are said to be *forward adjacent* (respectively *backward adjacent*). The following proposition is essentially due to Handel [Han99] (a proof will be given in section 3).

Proposition 2.1. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class. If $(\alpha_i^{\pm})_i$ and $(\beta_i^{\pm})_i$ are two nice families for this class, then they have the same cyclic order up to permutation of arcs of $(\beta_i^{\pm})_i$ inside the same sub-families of adjacency.*

2.1.3 Diagram associated to a Brouwer mapping class

Using proposition 2.1, we can associate a diagram to each Brouwer mapping class (see figures 11 and 12 for examples):

1. Let $[h; \mathcal{O}]$ be a Brouwer mapping class relatively to r orbits. Choose a nice family $(\alpha_i^{\pm})_{1 \leq i \leq r}$.

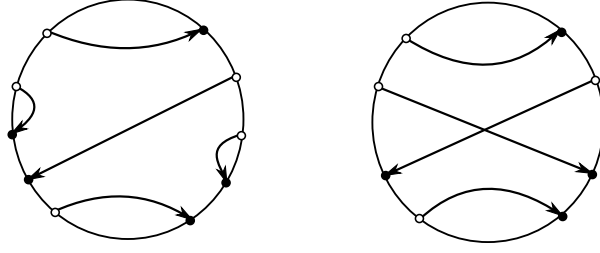


Figure 11: Diagrams associated to $[f; \mathcal{O}]$ and $[g; \mathcal{O}']$ (examples A and B of section 1).

2. On the boundary component of a disk, choose one point for each arc of $(\alpha_i^\pm)_i$ in such a way that the $2r$ chosen points respect the cyclic order of $(\alpha_i^\pm)_i$.
3. For every i , draw an arrow from the point representing α_i^- to the point representing α_i^+ . Label this arrow with i .
4. Exchange the points in a same sub-family of adjacency if necessary to eliminate as many crossings as possible between the arrows.

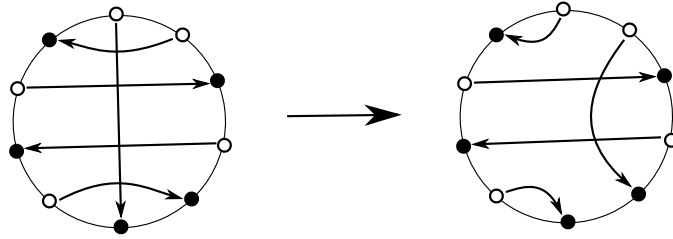


Figure 12: An example of step 4 (this is example 2.9 of Handel [Han99]).

We identify two diagrams if they have the same combinatorics.

Proposition 2.2. *The diagram associated to a Brouwer mapping class is a conjugacy invariant: if two Brouwer mapping classes are conjugated, then they have the same diagram.*

We say that a diagram \mathcal{D} is *determinant* if, up to conjugacy, there exists only one Brouwer mapping class whose associated diagram is \mathcal{D} . It is a natural question to ask which diagrams are determinant.

For Brouwer mapping classes relatively to one, two and three orbits, the diagram is a total invariant: every diagram with one, two or three arrows is determinant (see [Han99] for one and two orbits and [LR13] for three orbits).

However, for Brouwer mapping classes relatively to more than 3 orbits, the diagram is not a total invariant. For example, consider the flow f and the f -free half twist μ of examples A and B. The Brouwer mapping classes $[\mu^2 f; \mathcal{O}']$ and $[f; \mathcal{O}']$ have the same

diagram but are not conjugated (we will prove later that they are not conjugated). In section 5, we will give combinatorial conditions on diagrams to prove that some of them are determinant. In section 7 we will describe all the determinant diagrams for Brouwer mapping classes relatively to 4 orbits.

2.1.4 Special nice families

A *reducing set* is a union of mutually disjoint and non isotopic reducing lines. Any connected component of the complement of a reducing set is said to be a *stable area*. In particular, every stable area for $[h; \mathcal{O}]$ is isotopic to its image by h relatively to \mathcal{O} . If the reducing lines of a given reducing set \mathcal{R} are geodesic, then according to the straightening principle 1.1, there exists $h' \in [h; \mathcal{O}]$ such that every stable area Z of the complement of \mathcal{R} is such that $h'(Z) = Z$. The following proposition will be proved in section 3.3.

Proposition 2.3. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Let $(\Delta^k)_k$ be a reducing set. There exists a nice family $(\alpha_i^\pm)_i$ for $[h; \mathcal{O}]$ such that for every k , for every i , α_i^- and α_i^+ are homotopically disjoint from Δ^k .*

2.2 Walls for a Brouwer mapping class (section 4)

We define a canonical reducing set (the walls) and prove that this set splits the plane into three types of stable areas: stable areas disjoint from \mathcal{O} , translation areas and irreducible areas. In the next section, we will give combinatorial conditions for the existence of irreducible areas.

2.2.1 Translation areas

Definition 2.1 (Translation area). Let $[h; \mathcal{O}]$ be a Brouwer mapping class. We say that a stable area Z of $[h; \mathcal{O}]$ is a *translation area* if all the orbits of Z are backward adjacent and forward adjacent for $[h; \mathcal{O}]$.

Moreover, a translation area Z is said to be *maximal* if there exists no translation area Z' non isotopic to Z and such that $Z \subset Z'$.

Note that every translation area is included in a maximal translation area.

Remark 2.1. The orbits of a same translation area for a Brouwer mapping class are represented by arrows which are backward and forward adjacent in the diagram associated to the Brouwer class. However, some arrows which are backward and forward adjacent do not represent orbits of a translation area. For an example, see figure 13: the Brouwer class that we consider is the product of a flow with a double free half twist between the two arrows that intersect the grey disk of the figure. The two arrows on the bottom of the diagram are backward and forward adjacent but not in the same translation area.

The following proposition implies that every Brouwer class has flow streamlines on its translation areas. It will be proved in section 4.1.

Proposition 2.4. *If Z is a translation area, every backward (respectively forward) arc of a nice family which is included in Z is also forward (respectively backward).*

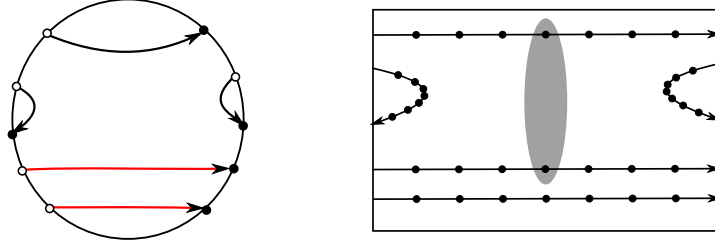


Figure 13: Example of a Brouwer class whose diagram has two backward and forward adjacent arrows which are not in the same translation area.

2.2.2 Irreducible areas

Definition 2.2 (Irreducible area). Let $[h; \mathcal{O}]$ be a Brouwer mapping class. We say that a stable area Z of $[h; \mathcal{O}]$ is an *irreducible area* if:

- Z contains at least 2 orbits of \mathcal{O} ;
- There is no reducing line of $[h; \mathcal{O}]$ strictly included in Z (i.e. homotopically disjoint from every boundary component of Z and non isotopic to any of those).

2.2.3 Walls

Definition 2.3 (Wall). Let $[h; \mathcal{O}]$ be a Brouwer mapping class. An isotopy class of a reducing line Δ for $[h; \mathcal{O}]$ is called a *wall* for $[h; \mathcal{O}]$ if every reducing line for $[h; \mathcal{O}]$ is homotopically disjoint from Δ .

The proof of the following theorem is the aim of section 4.

Theorem 2.5. Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Let \mathcal{W} be a family of pairwise disjoint reducing lines containing exactly one representative of each wall for $[h; \mathcal{O}]$. If Z is a connected component of $\mathbb{R}^2 - \mathcal{W}$, then exactly one of the followings holds:

- Z is an irreducible area;
- Z is a maximal translation area;
- Z does not intersect \mathcal{O} .

Remark 2.2. Note that:

- The set of walls is empty if and only if $[h; \mathcal{O}]$ is a translation class;
- There exists exactly one wall for $[h; \mathcal{O}]$ if and only if $[h; \mathcal{O}]$ is the class of a Reeb flow.

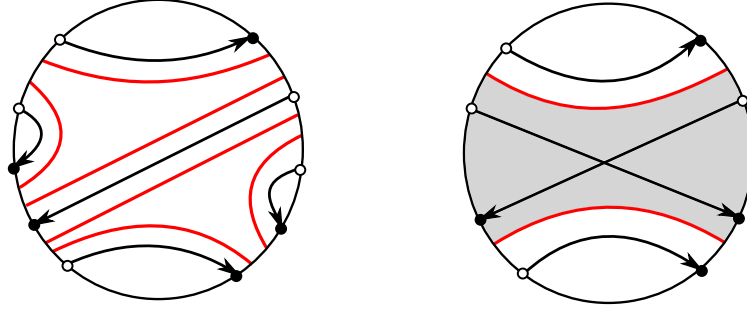


Figure 14: Diagrams with walls associated to $[f; \mathcal{O}]$ and $[g; \mathcal{O}']$ (examples A and B of section 1).

Since there exists a nice family disjoint from the walls (according to proposition 2.3), it makes sense to add the set of walls on the diagram defined in section 3 (see figure 14 for two examples). We can see the maximal translation areas on this diagram: the backward ends of their arrows are adjacent in the diagram and the forward ends of their arrows are adjacent in the diagram. Consequently, according to theorem 2.5, we can also see the irreducible areas. To help the reader, we color the irreducible areas in grey. The resulting *diagram with walls* is a conjugacy invariant of the Brouwer mapping class which is more precise than the diagram (without walls), but still not total. In the next section we will give conditions to determine which diagrams with walls are determinant.

2.3 Determinant diagrams and irreducible areas (section 5)

2.3.1 Determinant diagrams

The following propositions motivate the search of necessary combinatoric conditions on diagrams (without walls) for the existence of irreducible areas. They will be proved in section 5.1.

Proposition 2.6. *A Brouwer mapping class $[h; \mathcal{O}]$ is a flow class if and only if no connected component of the complement of the set of walls for $[h; \mathcal{O}]$ is an irreducible area.*

Proposition 2.7. *If two flow classes have the same diagram, then they are conjugated.*

We will deduce that a diagram without crossing arrows is determinant if and only if it does not have any irreducible area:

Proposition 2.8. *A diagram with walls without crossing arrows is determinant if and only if the arrows of every family of arrows included in the same connected component of the complement of the walls are backward adjacent and forward adjacent.*

2.3.2 Combinatorics of irreducible areas

If $[h; \mathcal{O}]$ is a Brouwer class relatively to r orbits, we denote by $2r'$ the number of adjacency subfamilies of $[h; \mathcal{O}]$. If $r' = r$, then we say that the orbits of $[h; \mathcal{O}]$ alternate (in this situation, every adjacency subfamily has only one element). We will prove proposition 2.9 in section 5.2.

Proposition 2.9 (Combinatorics of irreducible areas). *Let $[h; \mathcal{O}]$ be a Brouwer mapping class and let Z be an irreducible area for $[h; \mathcal{O}]$. Then:*

1. *The orbits of Z are not all backward adjacent, neither all forward adjacent for $[h; \mathcal{O}]$;*
2. *Z has at least two boundary components;*
3. *The orbits of $[h; \mathcal{O} \cap Z]$ do not alternate.*

This proposition gives tools to know which diagrams are determinant. In particular, we have the following corollaries, which will be proved in section 5.3:

Corollary 2.10. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class relatively to r orbits. Denote by $2r'$ the number of adjacency subfamilies of $[h; \mathcal{O}]$. If $r' = 1, 2$ or r , then $[h; \mathcal{O}]$ is a flow class.*

Remark 2.3. It was proved in [Han99] (for $r' = 1$) and [LR13] (for $r' = r$) that if $r' = 1$ or $r' = r$, then $[h; \mathcal{O}]$ is a flow class (see proposition 3.1 and lemma 3.6 of [LR13]).

Corollary 2.11. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class relatively to r orbits.*

1. *If $r \geq 3$, there exist at least two disjoint and non isotopic reducing lines for $[h; \mathcal{O}]$;*
2. *If $r \geq 2$, there exist at least two translation areas for $[h; \mathcal{O}]$ which have exactly one boundary component;*
3. *There exists a nice family $(\alpha_i^\pm)_i$ and $j \neq k$ such that:*
 - *Relatively to \mathcal{O} , α_j^- is isotopic to α_j^+ and α_k^- is isotopic to α_k^+ ;*
 - *In the cyclic order, α_j^- and α_j^+ (respectively α_k^- and α_k^+) are neighbors.*

The third point of corollary 5.5 can be reformulated as follow.

Corollary 2.12. *Let $[h; \mathcal{O}]$ be a Brouwer class relatively to $r \geq 2$ orbits. There exist at least two backward and forward disjoint proper streamlines T and S for $[h; \mathcal{O}]$.*

Moreover, the orbits which are not in T nor S are included in the same connected component of the complement of $S \cup T$.

2.4 Classification relatively to 4 orbits (section 7)

The aim of section 7 is to give a complete description of Brouwer mapping classes relatively to 4 orbits. We first find every diagram with walls which are not determinant. For the Brouwer mapping classes with a non determinant diagram, we define a new conjugacy invariant, *the tangle* (see section 7.2.2). This tangle is an isotopy class of curves on the cylinder with two marked points, up to horizontal twists (see figure 15 for an example). We set that the tangle of Brouwer mapping classes without irreducible area is the empty set. We claim that the couple (Diagram with walls, Tangle) is a total conjugacy invariant for Brouwer mapping classes relatively to 4 orbits:

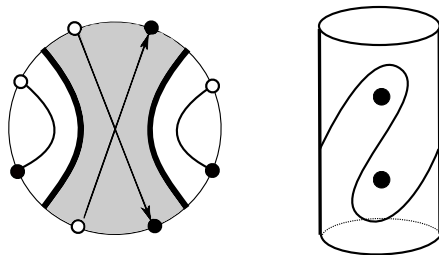


Figure 15: Example of a couple (Diagram with walls, Representative of the tangle)

Theorem 2.13. *Two Brouwer mapping classes relatively to 4 orbits are conjugated if and only if they admit the same couple (Diagram with walls, Tangle).*

We describe which tangles are realized by Brouwer mapping classes and call them *adapted tangles*.

Finally, every couple (Diagram with walls, Adapted tangle) is realized by:

- A flow if the diagram is determinant or if the tangle is trivial;
- A product of a flow and finitely many free half twists if the diagram is not determinant and the tangle is not trivial.

This gives a complete description of the Brouwer mapping classes relatively to 4 orbits.

3 Adjacency areas, diagrams and special nice families

3.1 Adjacency areas

Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Let $(\alpha_i \pm)_{1 \leq i \leq r}$ be a nice family. Let $\{\alpha_{i_1}^+, \dots, \alpha_{i_n}^+\}$ be a sub-family of adjacency for $[h; \mathcal{O}]$. For simplicity of notation, we assume that $i_k = k$ for every $1 \leq k \leq n$. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. Let L be a geodesic topological line in $\mathbb{R}^2 - \mathcal{O}$ such that (see figure 16):

- One connected component of $\mathbb{R}^2 - L$, denoted by A , contains an infinite component of each T_i^+ for $1 \leq i \leq n$;

- For every $1 \leq i \leq n$, L intersects $T_i^+ := T^+(\alpha_i^+, h, \mathcal{O})$ in exactly one point;
- A does not contain any point of \mathcal{O} outside those n half streamlines.

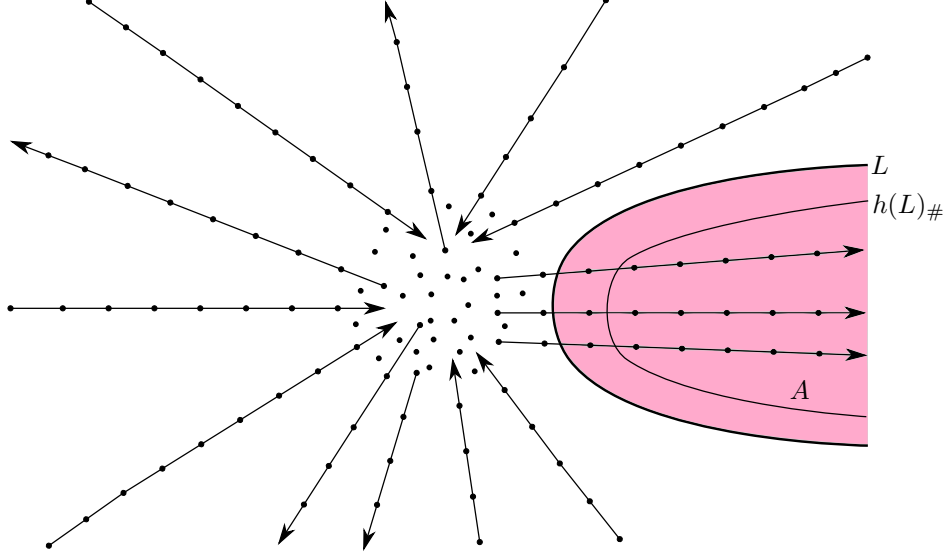


Figure 16: Example of a forward adjacency area.

Definition 3.1 (Adjacency area). With the previous notations, we say that A is a *forward adjacency area* for $[h; \mathcal{O}]$. A forward adjacency area for $[h^{-1}; \mathcal{O}]$ is said to be a *backward adjacency area* for $[h; \mathcal{O}]$.

Note that every adjacency area can be obtained with the following construction.

Construction of an adjacency area. With the previous notations, suppose that the α_i^+ 's for $i = 1 \dots n$ are ordered from α_1^+ to α_n^+ in the cyclic order of the nice family. Choose one point $x_i \in \mathcal{O}_i$ in every T_i^+ . Denote by L_i the unbounded component of $T_i^+ - \{x_i\}$.

- If $n = 1$, let \mathcal{U} be an open neighborhood of L_1 which is homotopic to L_1 relatively to \mathcal{O} ;
- If $n > 1$: for every $i \leq n - 1$, denote by γ_i a geodesic arc of $\mathbb{R}^2 - \mathcal{O}$ which admits x_i and x_{i+1} for endpoints, and such that one connected component of $L_i \cup \gamma_i \cup L_{i+1}$ does not intersect \mathcal{O} . In particular, note that $\{h^n(\gamma_i)_\# \}_{n \geq 0}$ is locally finite. Now consider the line $\tilde{L} := L_1 \cup \gamma_1 \cup \dots \cup \gamma_n \cup L_n$. Let \mathcal{V} be the connected component of $\mathbb{R}^2 - \tilde{L}$ which contains L_2 if $n \geq 3$ and which does not intersect \mathcal{O} if $n = 2$. Let \mathcal{U} be an open neighborhood of \mathcal{V} , isotopic to \mathcal{V} relatively to \mathcal{O} .

Assume that the boundary component L of the closure of \mathcal{U} is geodesic: then \mathcal{U} is an adjacency area. Since there exist pairwise disjoint half homotopy streamlines (theorem 1.2), there exist pairwise disjoint adjacency areas whose union contain an infinite component of every half orbit (see figure 17).

Definition 3.2. Choose an adjacency area for every subfamily of adjacency such that the chosen areas are mutually disjoint. Such a family is said to be a *complete family of adjacency areas* for $[h; \mathcal{O}]$.

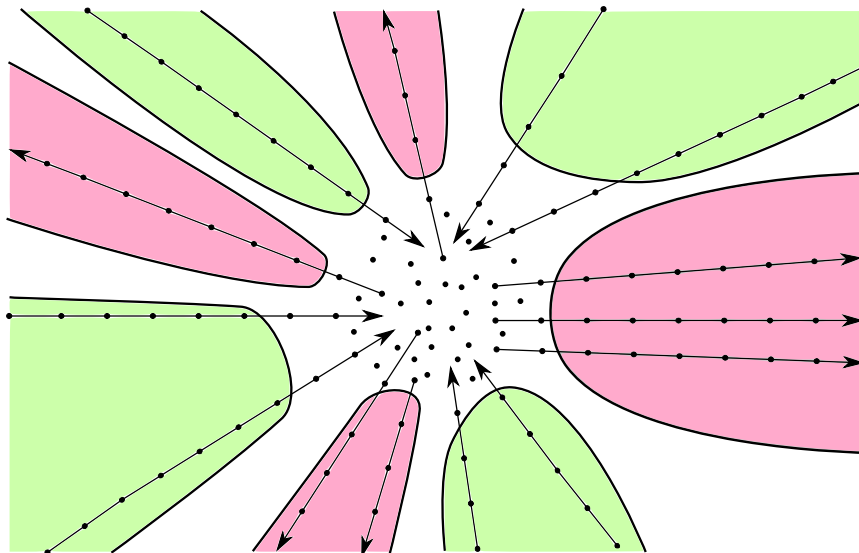


Figure 17: Example of a complete family of backward and forward adjacency areas.

In particular, the union of the elements of a complete family of adjacency areas contains every point of \mathcal{O} but a finite number. Moreover, if we consider two complete families of adjacency areas, then there exists a compact K such that these two families are isotopic relatively to \mathcal{O} in $\mathbb{R}^2 - K$.

The following theorem is essentially due to Handel [Han99]. This statement is a reformulation of proposition 3.1 of [LR13].

Theorem 3.1 (Handel). *Let $[h; \mathcal{O}]$ be a Brouwer class which is not a translation class. Choose a complete family of adjacency. There exists a reducing line disjoint from all the backward adjacency areas of the complete family.*

Proof. There exists a family of generalized homotopy half streamline such that for every backward adjacency area A of the complete family, $\mathcal{O} \cap A$ is included in one of the backward generalized homotopy half streamlines. Proposition 3.1 of [LR13] gives a reducing line disjoint from all the backward generalized homotopy half streamlines of the family. The result follows. \square

Proposition 3.2. *Let A be an adjacency area for $[h; \mathcal{O}]$ and let L be its boundary component.*

1. L is a homotopy Brouwer line;
2. The family $(h^k(L)_\#)_{k \in \mathbb{N}}$ is locally finite;

3. (Handel) Let β^+ be any forward proper arc with endpoints in an orbit of \mathcal{O} which intersects A . There exists k_0 such that for every $k > k_0$, $h^k(\beta^+)_{\#}$ is included in A .

Proof. Every adjacency area can be seen as in the construction done before. Thus (1) and (2) follow, because $\{h^n(\gamma_i)_{\#}\}_{n \geq 0}$ is locally finite for every i , as well as $\{h^n(L_i)_{\#}\}_{n \geq 0}$. The constructed \mathcal{U} is a neighborhood of a "generalized homotopy streamline" (see [Han99] and figure 18), hence property (3) holds, according to lemma 4.6 of [Han99]. \square

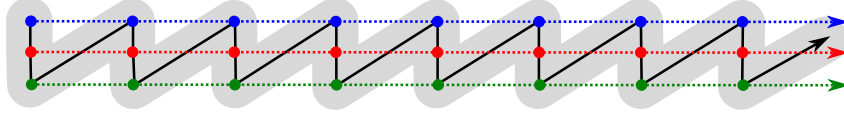


Figure 18: Neighborhoods of generalized homotopy half streamlines are homotopic to adjacency areas.

Corollary 3.3. Let $[h; \mathcal{O}]$ be a Brouwer mapping class, and let A be an adjacency area for $[h; \mathcal{O}]$. Then there exists a homeomorphism χ of the plane, preserving the orientation, such that $\chi h \chi^{-1}$ is isotopic relatively to $\chi(\mathcal{O})$ to a homeomorphism which coincides with a translation on $\chi(A)$.

Proof. This follows from (1) and (2) of proposition 3.2. \square

3.2 Diagrams

Proofs of propositions 2.1 and 2.2. Let $(\alpha_i^{\pm})_i$ and $(\beta_i^{\pm})_i$ be two nice families for a Brouwer class $[h; \mathcal{O}]$. According to (3) of proposition 3.2, for every i there exists an adjacency area A and there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$, $h^k(\alpha_i^{\pm})$ and $h^k(\beta_i^{\pm})$ are included in A . We have a similar result for backward arcs. It follows that $(\alpha_i^{\pm})_i$ and $(\beta_i^{\pm})_i$ have the same cyclic order up to permutation of arcs of $(\beta_i^{\pm})_i$ inside the same sub-families of adjacency, which is proposition 2.1.

As a corollary we get proposition 2.2: if two Brouwer mapping classes are conjugated, then they have the same diagram. \square

3.3 Special nice families

The aim of this section is to prove proposition 2.3, i.e. that for every reducing set \mathcal{R} , there exists a nice family which is disjoint from \mathcal{R} .

3.3.1 Intersections between reducing lines and adjacency areas

Lemma 3.4. Let $[h; \mathcal{O}]$ be a Brouwer mapping class, with a complete family of adjacency areas. Let Δ be a reducing line. Then Δ is isotopic relatively to \mathcal{O} to a topological line Δ' which intersects at most two adjacency areas of the family.

Moreover, for any complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$, the intersection between the geodesic representative of Δ for this metric and any adjacency area has at most finitely many connected components.

Proof. Let $(\alpha_i^\pm)_i$ be a nice family for $[h; \mathcal{O}]$. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. For every i , we denote by T_i^+ , respectively T_i^- , the homotopy half streamline $T^+(\alpha_i^+, h, \mathcal{O})$, respectively $T^-(\alpha_i^+, h, \mathcal{O})$. According to the straightening principle 1.1, there exists $h' \in [h; \mathcal{O}]$ such that $h'(T_i^+) \subset T_i^+$, $T_i^- \subset h'(T_i^-)$ and $h'(\Delta_\#) = \Delta_\#$.

Claim 1. Let A be an adjacency area. We denote by ∂A its boundary component. If $\Delta_\# \cap \partial A$ is non empty, then $\Delta_\# \cap h^n(\partial A)_\#$ is non empty for every $n \in \mathbb{Z}$.

Proof of claim 1. Since $\Delta_\# \cap \partial A$ is non empty, $h^n(\Delta_\#) \cap h^n(\partial A)$ is non empty for every $n \in \mathbb{Z}$. Since $\Delta_\#$ and ∂A are geodesic, they are in minimal position. Hence for every n , $h^n(\Delta_\#)$ and $h^n(\partial A)$ are also in minimal position. It follows that $h^n(\Delta_\#) \cap h^n(\partial A)_\#$ is non empty. \square

Claim 2. Let A be an adjacency area. If $\Delta_\# \cap A$ is non empty, then for every compact subset K of the plane, $(\Delta_\# \cap A) - K$ is non empty.

Proof of claim 2. Assume A is a forward adjacency area (if not, consider h'^{-1} instead of h). Let K be any compact subset of the plane. Assume that $\Delta_\# \cap A$ is non empty. Since $(h^n(\partial A)_\#)_{n \in \mathbb{N}}$ is locally finite (according to (2) of proposition 3.2), there exists $k \in \mathbb{N}$ such that $h'^k(\partial A)_\#$ does not intersect K . Since ∂A is a homotopy Brouwer line (according to (1) of proposition 3.2), $h'^k(\partial A)_\#$ is included in A . According to claim 1, $h'^k(\partial A)_\#$ intersects $\Delta_\#$. Claim 2 follows. \square

Denote by $(A_i)_{1 \leq i \leq l}$ the adjacency areas of the chosen complete family. According to claim 1, if we prove that for some $(n_i)_i \in \mathbb{Z}^l$, $\Delta_\#$ intersects at most two of the $h^{n_i}(\partial A_i)_\#$'s, then $\Delta_\#$ intersects at most two of the ∂A_i 's. Hence, up to replacing $(A_i)_{1 \leq i \leq l}$ by $(h^{n_i}(\partial A_i)_\#)_{1 \leq i \leq l}$ such that $\Delta_\#$ intersects at most two of the $h^{n_i}(\partial A_i)_\#$'s, we can assume that for every i, j , A_i is disjoint from α_j^- .

Claim 3. There exists a topological disk K of the plane such that every connected component of $\Delta_\# - K$ intersects at most one adjacency area.

Proof of claim 3. Assume by contradiction that for every K , one connected component of $\Delta_\# - K$ intersects two adjacency areas. Then there exist two adjacency areas, say A_i^- and A_j^+ , such that $\Delta_\# \cap A_i^-$ and $\Delta_\# \cap A_j^+$ have infinitely many connected components. Moreover, taking K which intersects every adjacency area of the complete family, we can suppose that A_i^- follows A_j^+ in the cyclic order at infinity of the adjacency areas. Hence we can suppose that A_i^- is a backward adjacency area, and A_j^+ is a forward adjacency area.

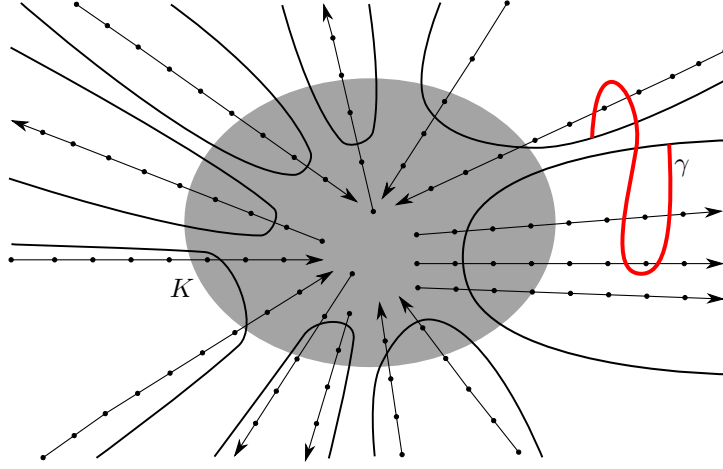


Figure 19: Example of a configuration with some K and some γ .

It follows that there exists a subsegment γ of $\Delta_{\#}$ such that γ is the concatenation of three smaller subsegments $\gamma_1 \star \gamma_2 \star \gamma_3$ such that (see figure 19):

- $\gamma_1 \subset A_i^-$ and its endpoints are included in ∂A_i^- ;
- $\gamma_3 \subset A_j^+$ and its endpoints are included in ∂A_j^+ ;
- γ_2 does not intersect \mathring{A}_i^- nor \mathring{A}_j^+ , where \mathring{A} denotes the interior $A - \partial A$ of A .

Moreover, we can choose γ outside any chosen topological disk of the plane: in particular, we choose it disjoint from the α_i^- 's. Since $\Delta_{\#}$ and the boundary components of the adjacency areas are in minimal position, it follows that:

- γ_1 intersects a backward half homotopy streamline T_i^- of A_i^- ;
- γ_2 does not intersect any half homotopy streamline of the family $(T_k^{\pm})_k$;
- γ_3 intersects a forward half homotopy streamline T_j^+ of A_j^+ .

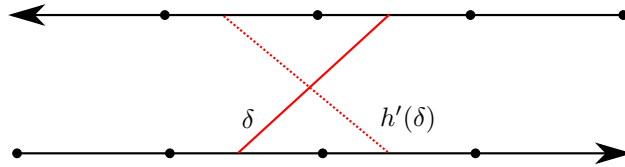


Figure 20: Image of δ under h' .

Hence there exists a subsegment δ of γ , which contains γ_2 , such that its endpoints are in T_i^- and T_j^+ but its interior does not intersect any T_k^{\pm} . Since h' acts as a translation

on the T_k^\pm 's, it follows that $h'(\delta)$ intersects δ . This gives a contradiction because $\Delta_\#$ is invariant by h' and without self-intersection (see figure 20). \square

Let K be a topological disk given by claim 3. Since $\Delta_\#$ is proper, there are only two unbounded connected components in $\Delta_\# - K$. According to claim 2, as $\Delta_\#$ and the boundary components of the adjacency areas are in minimal position, every connected component of $\Delta_\# - K$ which intersects an adjacency area is unbounded. Hence $\Delta_\#$ intersects at most two adjacency areas. Moreover, since $\Delta_\#$ and ∂A_k are geodesics, the second part of the lemma follows. \square

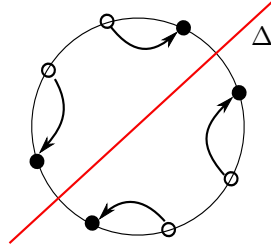


Figure 21: Example. Let $[f; \mathcal{O}]$ be a flow class with this diagram: the reducing line Δ intersects two forward adjacency areas.

Remark: If a reducing line intersects two adjacency areas, it does not necessary intersects one backward adjacency area and one forward adjacency area: some reducing lines intersect two adjacency areas of the same type (see figure 21 for an example).

The following lemma will be used in the proof of (2) of proposition 2.9.

Lemma 3.5. *Let $[h; \mathcal{O}]$ be a Brouwer class which is not a translation class. Choose a complete family of adjacency areas. There exists a reducing line Δ such that the intersection of Δ with the complement of the forward adjacency areas is bounded.*

Proof. Let $(\alpha_i^\pm)_i$ be a nice family such that each α_i^- intersects the boundary component of a backward adjacency area of the chosen family (such a family exists, up to take iterates of arcs of any nice family). Since $[h; \mathcal{O}]$ is not a translation class, according to theorem 3.1, there exists a reducing line Δ which is disjoint from every backward adjacency area. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. Suppose Δ is geodesic. Let K be a topological compact disk of the plane such that:

- The boundary component ∂K of K is geodesic;
- K intersects every adjacency areas of the family;
- The union \mathcal{A} of K with all the adjacency areas of the family contains \mathcal{O} .

Since Δ is proper, it intersects ∂K only finitely many times. Denote by Δ^+ and Δ^- the two unbounded components of $\Delta - K$. Since Δ^+ and Δ^- are disjoint from the backward adjacency areas, we can isotopy each of them if necessary to include Δ in a forward adjacency area. \square

3.3.2 Proof of proposition 2.3

Proposition. 2.3 *Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Let $(\Delta^k)_k$ be a reducing set. There exists a nice family $(\alpha_i^\pm)_i$ for $[h; \mathcal{O}]$ such that for every k , for every i , α_i^- and α_i^+ are homotopically disjoint from Δ^k .*

Proof. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. As usual, we denote by $L_\#$ the geodesic representative of any line or arc L . Choose a complete family of adjacency areas for $[h; \mathcal{O}]$. In every adjacency area, we will construct pairwise disjoint backward or forward proper arcs for each orbit which intersects the area, such that every constructed arc is disjoint from the reducing set. By iterating those arcs so that the backward and forward arcs of a given orbit have the same endpoints, we get the needed nice family. Let A be an adjacency area of the complete family of adjacency. Suppose A is a forward adjacency area (if not, consider h^{-1}). Note that ∂A is geodesic (by definition). For simplicity of notations, we assume that $\Delta^1, \dots, \Delta^N$ are the reducing lines of $(\Delta^k)_k$ intersecting A . We assume that these reducing lines are geodesic. According to lemma 3.4, each of them has an unbounded connected component included in A . Denote by Δ_+^k this unbounded component for Δ^k . Suppose that $\mathcal{O}_1, \dots, \mathcal{O}_m$ are the orbits of \mathcal{O} which intersect A . We will find m mutually disjoint forward proper arcs $(\beta_i^+)_{1 \leq i \leq m}$ for $[h; \mathcal{O}]$, included in A and homotopically disjoint from Δ^k for every k .

Applying the straightening principle 1.1 to the families $(\Delta^k)_k$ and $(h^n(\partial A))_{n \geq 0}$, we can find $h' \in [h; \mathcal{O}]$ such that $h'(\Delta^k) = \Delta^k$ for every k and $(h')^n(\partial A) = h^n(\partial A)_\#$ for every n . Note that h' is conjugate to the translation on A (according to corollary 3.3).

Let \mathcal{C} be the quotient of A by h' . Denote by π the quotient map. In particular:

- \mathcal{C} is a topological cylinder;
- $\pi(\mathcal{O} \cap A)$ is a set of m points $\hat{x}_1, \dots, \hat{x}_m$ on \mathcal{C} ;
- For every k , $\pi(\Delta^k \cap A) = \pi(\Delta_+^k)$ is a separating topological circle of $\mathcal{C} - \{\hat{x}_1, \dots, \hat{x}_m\}$;
- The $\pi(\Delta^k \cap A)$'s are mutually disjoint.

For simplicity of notation, we see \mathcal{C} as a vertical cylinder. There exists a homeomorphism ϕ of \mathcal{C} sending each $\pi(\Delta^k \cap A)$ on a horizontal circle γ_k and the family $(\hat{x}_i)_i$ on points of \mathcal{C} with mutually distinct heights. Now for every $1 \leq i \leq m$, consider the horizontal circle γ'_i containing $\phi(\hat{x}_i)$. Every γ'_i is disjoint from $\phi\pi(\Delta^k \cap A)$ for every k . Hence $(\phi^{-1}(\gamma'_i))_i$ is a family of mutually disjoint curves disjoint from $\pi(\Delta^k \cap A)$ for every k . For every i , choose a lift β_i^+ of $\phi^{-1}(\gamma'_i)$ included in A , i.e. an arc included in A and such that $\pi(\beta_i^+) = \phi^{-1}(\gamma'_i)$. Such a β_i^+ is a translation arc. Since $\{h'^n(\partial A)_\#\}_{n \geq 0}$ is locally finite (proposition 3.2), the β_i^+ 's are forward proper. They are disjoint from the Δ^k 's, as wanted. \square

4 Walls for a Brouwer mapping class

The main aim of this section is to prove that the set of walls splits \mathbb{R}^2 into translation areas, irreducible areas and stable areas that do not intersect \mathcal{O} (theorem 2.5).

4.1 Translation areas

Lemma 4.1. *Let Z be a stable area of a Brouwer class such that all the orbits of Z are forward adjacent. Then every backward proper arc included in Z is forward proper.*

Proof. Let $[h; \mathcal{O}]$ be a Brouwer class with a complete family of adjacency areas. Let Z be a stable area for $[h; \mathcal{O}]$ which intersects only one forward adjacency area. Denote by A this adjacency area. Up to replace h by $h' \in [h; \mathcal{O}]$, according to the straightening principle 1.1, we can assume that $h(Z) = Z$. Let $(\alpha_i^\pm)_i$ be a nice family for $[h; \mathcal{O}]$ disjoint from the boundary components of Z (such a family exists, according to proposition 2.3). We prove the following claim, which is a consequence of theorem 5.5 of Handel [Han99].

Claim. For every i such that α_i^- is in Z , there exists n such that $h^n(\alpha_i^-)_\#$ is in A .

We use the definitions and notations of Handel [Han99], section 5 ("Fitted family"). We denote by W the Brouwer subsurface $\mathbb{R}^2 - \cup_k A_k^+$, where $\cup_k A_k^+$ is the union of forward adjacency areas. If $\alpha_i^- \in Z$ is such that for every $n \geq 0$, $h^n(\alpha_i^-)_\# \cap W \neq \emptyset$, then there exists a fitted family $T \subset RH(W, \delta_+ W)$ such that:

- Every $s \in T$ is included in Z (because the elements of T are subsegments of iterates of α_i^- , which is included in Z , and we have $h(Z) = Z$);
- There exists $t \in T$ whose endpoints lie on distinct components of $\delta_+ W$ (this is theorem (5.5.c) of [Han99]).

Since $\delta_+ W \cap Z$ has only one component (the boundary component of A), the last point does not hold, and thus every $\alpha_i^- \in Z$ is such that for every sufficiently big n , $h^n(\alpha_i^-)$ is homotopically included in A . It follows that every $\alpha_i^- \in Z$ is forward proper. \square

Proposition. 2.4 *If Z is a translation area, every backward (respectively forward) arc of a nice family which is included in Z is also forward (respectively backward).*

Proof. By definition, all the orbits of a translation area are backward adjacent and forward adjacent. The result is a consequence of lemma 4.1 applied to the Brouwer class, and respectively to its inverse. \square

4.2 Intersections between reducing lines

Lemma 4.2 (Intersection of two reducing lines). *Let $[h; \mathcal{O}]$ be a Brouwer mapping class and let Δ and Δ' be two reducing lines for $[h; \mathcal{O}]$. We assume that Δ and Δ' are in minimal position. Then one of the following situation holds.*

1. $\Delta \cap \Delta' = \emptyset$.
2. $\Delta \cap \Delta'$ contains exactly one point.
3. $\Delta \cap \Delta'$ is an infinite set.

Proof. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. Taking their images by isotopies if necessary, we can suppose that Δ and Δ' are geodesic. We use the straightening principle 1.1 to find a homeomorphism $h' \in [h; \mathcal{O}]$ such that h' preserves Δ and Δ' .

If $\Delta \cap \Delta'$ contains more than one point, then there exists a bounded connected component of $\mathbb{R}^2 - (\Delta \cup \Delta')$ which contains one point $x \in \mathcal{O}$. Denote by C_x this component. Then $h'(C_x)$ is a bounded component of $\mathbb{R}^2 - (\Delta \cup \Delta')$ different from C_x . Indeed, if $h'(C_x)$ coincides with C_x , then $h^n(C_x) = C_x$ for every $n \geq 0$. Hence $\{h^n(x)\}_{n \geq 0}$ is included in C_x . This is not possible because $h^n(x) = h^n(x)$ for every n : since h is a Brouwer homeomorphism, $\{h^n(x)\}_{n \geq 0}$ is not bounded (proposition 3.5 of [Gui94]).

For the same reasons, for every $k < n \in \mathbb{N}$, $h^n(C_x)$ is disjoint from $h^k(C_x)$. Thus there exist infinitely many bounded connected component of $\mathbb{R}^2 - (\Delta \cup \Delta')$. Hence Δ and Δ' have infinitely many intersection points. \square

Lemma 4.3 (Intersection between a reducing line and a translation area). *Let $[h; \mathcal{O}]$ be a Brouwer mapping class and let Z be a translation area for $[h; \mathcal{O}]$. Let Δ be a reducing line. If there exists a boundary component L of Z such that L and Δ are not homotopically disjoint, then $\Delta \cap L$ is an infinite set.*

Proof. The line L is a reducing line, hence it is isotopic to its image by h . Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. Suppose that L and Δ are geodesic. For every orbit \mathcal{O}_i of \mathcal{O} included in Z , we choose a homotopic proper translation arc α_i included in Z such that the α_i 's are mutually disjoint (given by proposition 2.3). If α is one of these homotopy translation arcs, we denote by T_α the proper streamline $\cup_{n \in \mathbb{Z}} h^n(\alpha)_\#$. Since L and Δ are not homotopically disjoint, there exists α_i such that $T_{\alpha_i} \cap \Delta \neq \emptyset$. Since T_{α_j} and L are disjoint for every j , the straightening principle 1.1 gives us a homeomorphism $h' \in [h; \mathcal{O}]$ which preserves L , Δ and T_{α_j} for every i .

Suppose that $\Delta \cap L$ is not infinite. According to lemma 4.2, since this intersection is not empty, it contains only one point, say x . In particular, we have $h'(x) = x$. Choose an orientation on Δ . Let y be the first intersection point between Δ and T_{α_i} after x on Δ . Denote by $[xy]$ the segment of Δ between x and y . We have $h'(y) \in T_{\alpha_i}$ and $h'([xy]) \cap (L \cup T_{\alpha_i}) = \emptyset$, hence $y = h'(y)$. This gives a contradiction because y is contained in a proper translation arc for h' . \square

Lemma 4.4 (Intersection between reducing lines: infinite set case). *Let $[h; \mathcal{O}]$ be a Brouwer mapping class and let Δ and Δ' be two reducing lines for $[h; \mathcal{O}]$. We assume that Δ and Δ' are in minimal position. If $\Delta \cap \Delta'$ is an infinite set, then $\Delta \cup \Delta'$ is included in a translation area.*

Proof. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. Isotopying if necessary, we can assume that Δ and Δ' are geodesic. The straightening principle 1.1 gives us $h' \in [h; \mathcal{O}]$ which preserves Δ and Δ' . Choose a complete family of mutually disjoint adjacency areas for $[h; \mathcal{O}]$. Choose a bounded connected component C_x of the complement of $\Delta \cup \Delta'$ which contains a point x of an orbit \mathcal{O}_i of \mathcal{O} . Denote by A^- and A^+ the backward and forward adjacency areas of the chosen complete family which are intersected by \mathcal{O}_i . As shown in the proof of lemma 4.2, $h'(C_x)$ is a bounded connected component of $\mathbb{R}^2 - (\Delta \cup \Delta')$ different from C_x . Hence every path from x to $h'(x)$ intersects $\Delta \cup \Delta'$.

According to proposition 2.3, there exists a forward proper arc α^+ for \mathcal{O}_i which joins x to $h'(x)$ and which is disjoint from Δ' . Denote by $T^+(\alpha^+)$ the forward half streamline $\cup_{n \geq 0} h^n(\alpha^+)_{\#}$. Note that $T^+(\alpha^+)$ is disjoint from Δ' . According to proposition 3.2, there exists an unbounded component of $T^+(\alpha^+)$ which is included in A^+ . Since $T^+(\alpha^+)$ is proper and disjoint from Δ' , the straightening principle 1.1 give us $h_1 \in [h; \mathcal{O}]$ which preserves $T^+(\alpha^+)$, Δ and Δ' . The arc α^+ intersects Δ , hence $h_1^n(\alpha^+)$ also intersects Δ for every $n \in \mathbb{N}$. It follows that Δ intersects A^+ .

The same argument with a backward proper arc α^- disjoint from Δ' shows that Δ also intersects A^- . According to lemma 3.4, every geodesic reducing line intersects at most two adjacency areas: for Δ , this adjacency areas are A^- and A^+ . Interchanging Δ and Δ' , we get by the same arguments that Δ' also intersects A^- and A^+ .

We choose an orientation on Δ and Δ' such that they are oriented from A^- to A^+ . There exists an unbounded connected component of the complementary of $\Delta \cup \Delta'$ which is on the left of Δ and Δ' . We denote by L_l its boundary component. Likewise, we denote by L_r the boundary component of the unbounded connected component of the complementary of $\Delta \cup \Delta'$ which is on the right of Δ and Δ' . The two lines L_l and L_r are proper, because they are unions of segments of two topological lines. Moreover, they are preserved by h' .

Now we have the following cases, depending on the positions of the orbits:

- If L_r and L_l split the set of orbits, then their geodesic representatives $(L_r)_{\#}$ and $(L_l)_{\#}$ are disjoint reducing lines which intersect the same adjacency areas. Denote by Z the stable area bounded by $(L_r)_{\#}$ and $(L_l)_{\#}$. Thus Z intersects only two adjacency areas, A^- and A^+ , hence Z is a translation area, which contains Δ and Δ' ;
- If none of L_r and L_l split the set of orbits, then there exist only two adjacency areas. Hence $[h; \mathcal{O}]$ is a translation, and the whole plane is a translation area;
- If only one of L_r and L_l splits the set of orbits, L_r for example, then L_r is a reducing line for $[h; \mathcal{O}]$. The connected component of $\mathbb{R}^2 - (L_r)_{\#}$ which contains Δ and Δ' is a translation area.

□

Lemma 4.5 (Intersection between reducing lines: case with exactly one point). *Let $[h; \mathcal{O}]$ be a Brouwer mapping class and let Δ and Δ' be two reducing lines for $[h; \mathcal{O}]$. We assume that Δ and Δ' are in minimal position.*

If $\Delta \cap \Delta'$ contains exactly one point, then there exist four reducing lines that are mutually non isotopic and homotopically disjoint and disjoint from Δ and Δ' .

Moreover, if we denote by p the intersection point and by Δ_1 and Δ_2 , respectively Δ'_1 and Δ'_2 , the two half-lines of $\Delta - \{p\}$, respectively of $\Delta' - \{p\}$, these four reducing lines are isotopic relatively to \mathcal{O} to $\Delta_1 \cup \Delta_2$, $\Delta'_1 \cup \Delta'_2$, $\Delta_1 \cup \Delta'_1$ and $\Delta_2 \cup \Delta'_2$.

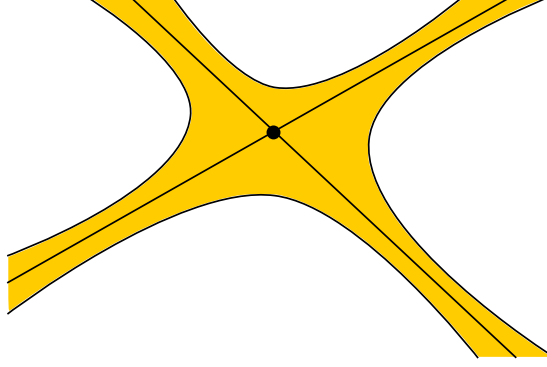


Figure 22: Neighborhood of $\Delta \cup \Delta'$.

Proof. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. Isotopying if necessary, we can assume that Δ and Δ' are geodesic. The straightening principle 1.1 gives us $h' \in [h; \mathcal{O}]$ which preserves Δ and Δ' .

Consider a proper open neighborhood \mathcal{U} of $\Delta \cup \Delta'$ which does not intersect \mathcal{O} and which is isotopic to $\Delta \cup \Delta'$ relatively to \mathcal{O} (as in figure 22). The complement of \mathcal{U} has 4 connected components. Each of them contains at least one orbit, because $\Delta \cup \Delta'$ are in minimal position. Hence the boundary component of the closure of \mathcal{U} in \mathbb{R}^2 is a union of 4 reducing lines, mutually non isotopic, mutually disjoint, and each of them is disjoint from $\Delta \cup \Delta'$. \square

4.3 Study of the set of walls

4.3.1 Maximal translation areas

We show that there exist finitely many maximal translation areas (proposition 4.6), and that the boundary components of this areas are walls (proposition 4.7).

Proposition 4.6. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Up to isotopy, there exist finitely many maximal translation areas. Moreover, they are mutually homotopically disjoint.*

Remark 4.1. The statement 4.6 is generally false if we replace *maximal translation area* by *translation area*. Indeed, if a translation area Z of a Brouwer class $[h; \mathcal{O}]$ contains at least two orbits, then there are infinitely many non isotopic sub-translation areas included in Z . See figure 23 for examples of reducing lines for the translation.

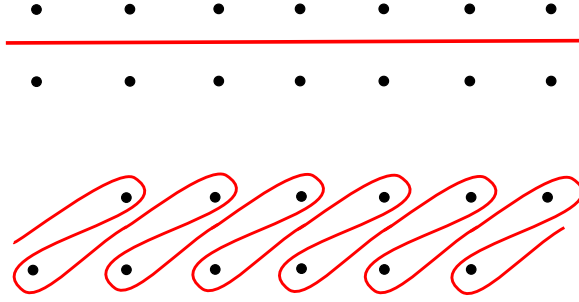


Figure 23: Example of two reducing lines for the translation: the complement are translation areas.

Proof of proposition 4.6. We choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$ and a complete family of adjacency areas for $[h; \mathcal{O}]$. Let Z and Z' be two non homotopic maximal translation areas. Denote by A^-, A^+ , respectively B^-, B^+ , the adjacency areas intersected by Z , respectively Z' . We claim that:

1. The boundary components of Z and Z' are homotopically disjoint;
2. No boundary component of Z is included in Z' .

Proof of (1). If a boundary component L of Z intersects a boundary component L' of Z' , then L and L' have infinitely many intersection points (according to lemma 4.3). Hence L and L' are included in the same translation area (according to lemma 4.4). Denote by Z'' this translation area, and by C^- and C^+ the two adjacency areas intersected by Z'' . Since L and L' are reducing lines, they intersect at most two adjacency areas: C^- and C^+ . The cyclic order of the adjacency areas at infinity is such that backward areas and forward areas alternate (by definition of adjacency areas). It follows that $A^- = B^- = C^-$ and $A^+ = B^+ = C^+$.

According to lemma 1.1, there exists $h' \in [h; \mathcal{O}]$ such that $h'(Z) = Z$ and $h'(Z') = Z'$. It follows that the boundary components of $Z \cup Z'$ are reducing lines, hence $Z \cup Z'$ is a stable area. Since $Z \cup Z'$ intersects only two adjacency areas (C^- and C^+), it is a translation area. This gives a contradiction with the maximality of Z and Z' as translation areas.

Proof of (2). If a boundary component L of Z is included in Z' , then there exists an orbit \mathcal{O}_i included in $Z \cap Z'$. Hence again $A^- = B^-$ and $A^+ = B^+$, which gives a contradiction.

We complete the proof of proposition 4.6. Every maximal translation area contains at least one orbit. Since there are finitely many orbits and since these areas are mutually disjoint, there are finitely many maximal translation areas. \square

Proposition 4.7. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class and let Z be a maximal translation area. Each isotopy class of a boundary component of Z is a wall of $[h; \mathcal{O}]$.*

Proof. Let L be a boundary component of a maximal translation area Z . We need to show that if Δ is a reducing line which is non isotopic to L , then $\Delta \cap L = \emptyset$. According to lemma 4.3, if a reducing line Δ intersects L then $\Delta \cap L$ is an infinite set. According to lemma 4.4, it follows that $L \cup \Delta$ is included in a translation area. Since maximal translation areas are mutually disjoint (according to proposition 4.6), Δ is included in Z (which contains L): this is impossible, hence every reducing line Δ is homotopically disjoint from L . \square

4.3.2 Outside the translation areas

This subsection completes the picture: there are finitely many maximal translation areas, mutually homotopically disjoint, and outside those areas there are only finitely many geodesic reducing lines, which intersect mutually in zero or one point.

Lemma 4.8. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Let Δ and Δ' be two disjoint reducing lines. If Δ and Δ' split the orbits into the same two subfamilies, then Δ and Δ' are isotopic.*

Proof. The set $\Delta \cup \Delta'$ splits the plane into three connected components. One of them (the one in the middle) is disjoint from \mathcal{O} . \square

Proposition 4.9. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Outside the maximal translation areas, there exists only finitely many non isotopic reducing lines.*

Proof. We prove that there exists only finitely many non isotopic reducing lines in every connected components of the complement of the union of the translation areas. If two such reducing lines are not homotopically disjoint, then they have only one intersection point (according to lemmas 4.2 and 4.4). Hence they do not split the orbits in the same subfamilies. This remark together with lemma 4.8 imply that if we choose a partition of the orbits in the chosen component into two subfamilies, there exists at most one reducing line included in the complement which splits the orbits into the same partition. Since there exist only finitely many different partitions of the orbits into two subfamilies, there are only finitely many isotopy classes of reducing lines. \square

4.3.3 Proof of theorem 2.5

Theorem. 2.5 *Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Let \mathcal{W} be a family of pairwise disjoint reducing lines containing exactly one representative of each wall for $[h; \mathcal{O}]$. If Z is a connected component of $\mathbb{R}^2 - \mathcal{W}$, then exactly one of the followings holds:*

- Z is an irreducible area;
- Z is a maximal translation area;
- Z does not intersect \mathcal{O} .

Proof. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. Up to isotopying relatively to \mathcal{O} , we can assume that the elements of \mathcal{W} are geodesic. According to proposition 4.7, the isotopy classes of boundary components of maximal translation areas are walls. Let Z be a connected component of $\mathbb{R}^2 - \mathcal{W}$ which is not a translation area. Suppose that Z is not irreducible. Then there exists a reducing line included in Z which is not homotopic to any boundary component of Z . According to proposition 4.9, Z contains a finite number of mutually non isotopic reducing lines. Since they are not walls, each of them intersects another one. In particular there are at least two reducing lines included in Z and not homotopic to any boundary component of Z .

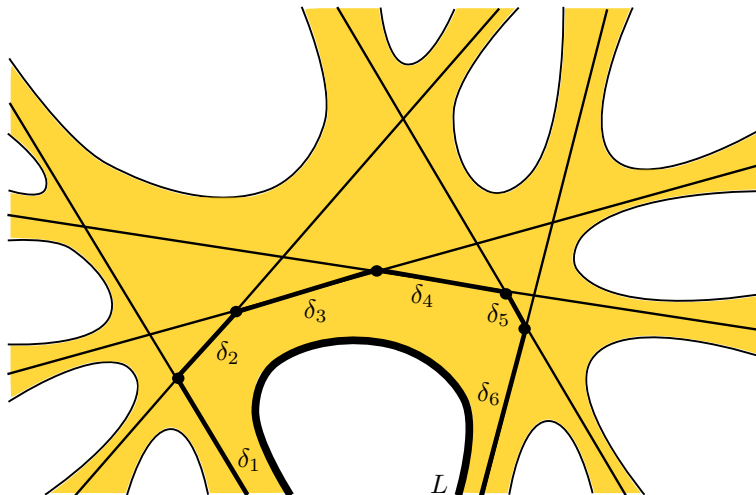


Figure 24: Example of \mathcal{U} , $\tilde{\mathcal{U}}$ and a boundary component L isotopic to $\delta_1 \cup \dots \cup \delta_6$.

Denote by \mathcal{U} the finite union of geodesic reducing lines included in Z and not homotopic to any boundary component of Z . Denote by $\tilde{\mathcal{U}}$ a tubular neighborhood of \mathcal{U} which does not intersect \mathcal{O} (see figure 24). Choose $\tilde{\mathcal{U}}$ such that the boundary components its closure are geodesic. Denote by L one of them. We claim the following:

Claim. The line L is a reducing line for $[h; \mathcal{O}]$.

Proof of the claim. The line L is homotopic to a union L' of a finite number of segments included in distinct reducing lines (see figure 25). The number of segments is finite because of the following properties.

1. The area Z is homotopically disjoint from the translation areas;
2. Up to isotopy there are only finitely many reducing lines outside the translation areas (according to proposition 4.9);
3. If two reducing lines outside the translation areas intersect, then their intersection is exactly one point: according to lemma 4.2, this intersection is either one point or

infinite, and according to lemma 4.4, if the intersection is infinite then the reducing lines are included in a translation area.

Denote by n this number, and by $\delta_1 \cup \dots \cup \delta_n$ the segments whose union is L' . We assume that the δ_i 's are in this order on L (as in figure 24). For every i , denote by Δ_i a reducing line of \mathcal{U} which contains δ_i . Denote by L_1 the line obtained as the union of δ_1 and the half-line of Δ_2 whose endpoint is the intersection point between δ_1 and δ_2 and which contains δ_2 . According to lemma 4.5, L_1 is a reducing line for $[h; \mathcal{O}]$. For every $2 \leq i \leq n-1$, denote inductively by L_i the line obtained as the union of the half-line L_{i-1} which contains δ_1 and the half-line Δ_{i+1} which contains δ_{i+1} (both half lines have the intersection point between δ_i and δ_{i+1} for endpoint). Applying lemma 4.5 inductively, we see that L_i is a reducing line for every i . Hence $L' = L_{n-1}$ is a reducing line for $[h; \mathcal{O}]$. \square

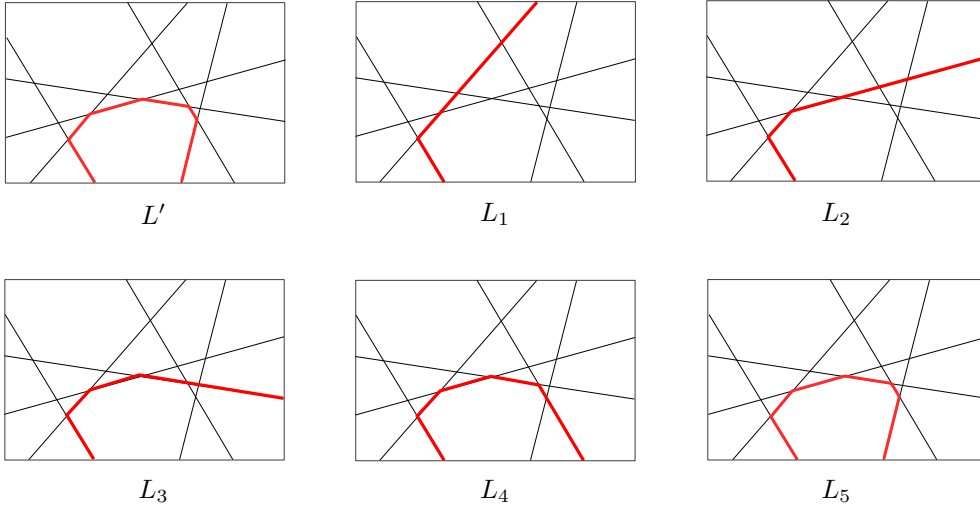


Figure 25: Proof of the claim in the case of the example.

End of the proof of theorem 2.5. Since L is in Z but not in \mathcal{U} , it is homotopic to a boundary component of Z (by definition of \mathcal{U}). Hence $\partial\tilde{\mathcal{U}}$ is included in ∂Z , thus Z is included in $\tilde{\mathcal{U}}$ (because they both are intersections of half topological planes). It follows that Z is isotopic to $\tilde{\mathcal{U}}$ relatively to \mathcal{O} , hence Z does not intersect \mathcal{O} . \square

Remark 4.2. Note that a stable area is irreducible if and only if it is an irreducible area of the complement of the set of walls. In particular, the isotopy classes of the boundary components of any irreducible area are walls.

5 Determinant diagrams and irreducible areas

5.1 Determinant diagrams

We prove here propositions 2.6, 2.7 and 2.8. We use the two following lemmas of [LR13] in the proofs:

Lemma 5.1 (Le Roux [LR13], lemma 1.8). *Let \mathcal{F} be a finite family of pairwise disjoint topological lines in the plane. Let h_0 be an orientation preserving homeomorphism of the plane. Let H_{h_0} be the space of orientation preserving homeomorphisms that coincide with h_0 on the union of the elements of \mathcal{F} . Then H_{h_0} is arcwise connected.*

Lemma 5.2 (Le Roux [LR13], lemma 1.6). *The Brouwer mapping class $[h; \mathcal{O}]$ is a fixed point free flow class if and only if it admits a family of pairwise disjoint proper geodesic homotopy streamlines whose union contains \mathcal{O} .*

Proposition. 2.6 *A Brouwer mapping class $[h; \mathcal{O}]$ is a flow class if and only if no connected component of the complement of the set of walls for $[h; \mathcal{O}]$ is an irreducible area.*

Proof. If $[f; \mathcal{O}]$ is a flow class, then according to lemma 5.2 we can choose a family of pairwise disjoint proper geodesic homotopy streamlines whose union contains \mathcal{O} . We find reducing lines in the neighborhood of each streamline, hence there is no irreducible area.

We now prove that if the set of walls \mathcal{W} of a Brouwer mapping class $[h; \mathcal{O}]$ is such that no component of the complement of \mathcal{W} is irreducible, then it is a flow class. According to proposition 2.3, there exists a nice family $(\alpha_i^\pm)_i$ for $[h; \mathcal{O}]$ disjoint from the walls. According to theorem 2.5, every connected component of the complement of \mathcal{W} which contain orbits is a translation area. According to proposition 2.4, every backward proper which is included in a translation area is also forward proper: it follows that every $T(\alpha_i^-, h, \mathcal{O})$ is a proper streamline. Lemma 5.2 gives us the conclusion. \square

Proposition. 2.7 *If two flow classes have the same diagram, then they are conjugated.*

Proof. Let $[f; \mathcal{O}]$ and $[g; \mathcal{O}']$ be two flow classes with the same diagram. According to lemma 5.2, there exists a nice family $(\alpha_i^\pm)_i$ for $[f; \mathcal{O}]$ and a nice family $(\beta_i^\pm)_i$ for $[g; \mathcal{O}']$ such that for every i , α_i^- is isotopic to α_i^+ , and β_i^- is isotopic to β_i^+ . We set $\alpha_i := \alpha_i^- = \alpha_i^+$ and $\beta_i := \beta_i^- = \beta_i^+$. Since $[f; \mathcal{O}]$ and $[g; \mathcal{O}']$ have the same diagram, $(\alpha_i^\pm)_i$ and $(\beta_i^\pm)_i$ have the same cyclic order at infinity (we permute the numbering of the orbits of \mathcal{O}' if necessary). Thus the Schoenflies theorem provides a homeomorphism of the plane which sent $T(\alpha_i, h, \mathcal{O})$ to $T(\beta_i, h, \mathcal{O})$ for every i . Lemma 5.1 gives the conclusion. \square

Proposition. 2.8 *A diagram with walls without crossing arrows is determinant if and only if the arrows of every family of arrows included in the same connected component of the complement of the walls are backward adjacent and forward adjacent.*

Proof. Let D be a diagram with walls without crossing arrows. Suppose D is determinant. Since D is without crossing arrows, there exists a flow class $[f; \mathcal{O}]$ whose associated diagram is D (this is lemma 1.7 of [LR13]). Since $[f; \mathcal{O}]$ is a flow class, every orbit of \mathcal{O} is included in a translation area, hence in a maximal translation area. In this situation, theorem 2.5 imply that every connected component of the complement of walls which contains orbit is a maximal translation area. The result follows.

If a diagram with walls D is such that every family of arrows included in the same connected component of the complement of the walls are backward adjacent and forward adjacent, then for every $[h; \mathcal{O}]$ whose associated diagram is D , the complement of the set of walls in \mathbb{R}^2 has no irreducible areas (it has only translation areas and areas without orbits). According to proposition 2.6, $[h; \mathcal{O}]$ is a flow class. If $[h'; \mathcal{O}']$ is another Brouwer class whose associated diagram with walls is D , then $[h'; \mathcal{O}']$ is also a flow class. It follows from proposition 2.7 that $[h; \mathcal{O}]$ and $[h'; \mathcal{O}']$ are conjugated. Hence the diagram with walls D is determinant. \square

5.2 Combinatorics of irreducible areas

We first prove a criterion for reducing lines and then use it to prove proposition 2.9.

5.2.1 A criterion for reducing lines

Lemma 5.3. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class and let $(\alpha_i^\pm)_i$ be a nice family for $[h; \mathcal{O}]$. If Δ is a topological line of $\mathbb{R}^2 - \mathcal{O}$ such that:*

1. Δ is a topological line;
2. Both components of $\mathbb{R}^2 - \Delta$ contain points of \mathcal{O} ;
3. For every i , for every $k \in \mathbb{Z}$, Δ is homotopically disjoint from $h^k(\alpha_i^-)$ relatively to \mathcal{O} .

Then Δ is a reducing line for $[h; \mathcal{O}]$.

Proof. We need to show that Δ and $h(\Delta)$ are isotopic relatively to \mathcal{O} . Choose a hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. Taking its image by an isotopy relatively to \mathcal{O} if necessary, we can assume that Δ is geodesic. We denote by f a representative of $[h; \mathcal{O}]$ mapping Δ on $h(\Delta)_\#$. Such an f exists, again according to the straightening principle 1.1. Hence Δ and $f(\Delta)$ are geodesic. We need to show that $\Delta = f(\Delta)$. Suppose that $\Delta \neq f(\Delta)$. We know that this two streamlines are in minimal intersection position (because they are geodesic), and we study separately the three possible cases: either Δ and $f(\Delta)$ have several intersection points, either they have only one intersection point, or they do not intersect. Those three cases lead us to contradictions.

If Δ and $f(\Delta)$ have several intersection points. We consider a subsegment γ_1 of $f(\Delta)$, whose endpoints are intersection points between Δ and $f(\Delta)$, denoted by a_1 and b_1 , and such that the open segment γ_1 is disjoint from Δ (see figure 26). Denote by δ_1 the subsegment of Δ between a_1 and b_1 . Since δ_1 is compact, it contains finitely

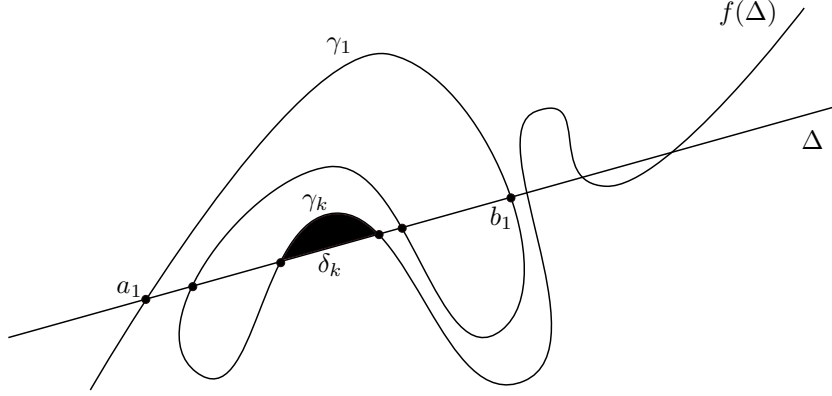


Figure 26: Example where Δ and $f(\Delta)$ have several intersection points.

many intersection points between Δ and $f(\Delta)$. Denote by n the number of intersections between Δ and $f(\Delta)$. Let us show that we can assume that $n = 0$. If $n > 0$, choose a_2 an intersection point between δ_1 and $f(\Delta)$. Consider the half line $f(\Delta)^+$ defined as the connected component of $f(\Delta) - a_2$ which has a subsegment with endpoint a_2 and which is included in the topological disk bounded by $\gamma_1 \cup \delta_1$. Because $f(\Delta)^+$ is proper, it goes out the topological disk bounded by $\gamma_1 \cup \delta_1$. Hence it intersects again δ_1 , because $f(\Delta)$ is without self intersection. Denote by b_2 the first intersection point between $f(\Delta)^+$ and δ_1 . The subsegments δ_2 and γ_2 of Δ and $f(\Delta)$ with endpoints a_2 and b_2 have the same properties than δ_1 and γ_1 , but the number of intersection points between Δ and $f(\Delta)$ on δ_2 is less than the one on δ_1 . Hence by recurrence there exists k and two subsegments $\delta_k \subset \Delta$ and $\gamma_k \subset f(\Delta)$ with endpoints a_k and b_k , and such that $\delta_k \cap f(\Delta) = \gamma_k \cap \Delta = a_k \cup b_k$.

Denote by D the topological disk bounded by $\delta_k \cup \gamma_k$. We claim that D does not intersect \mathcal{O} : if it contains a point of an orbit \mathcal{O}_i , denote by x_i this point. Let $n \in \mathbb{Z}$ be such that x_i is an endpoint of $h^n(\alpha_i^-)$. The family $(h^k(\alpha_i^-))_{k \leq n}$ is proper, because α_i^- is backward proper. Applying again the straightening principle 1.1, we find a homeomorphism g isotopic to h , which maps Δ on $h(\Delta)_\# = f(\Delta)$ and $h^k(\alpha_i^-)$ on $(h^{k+1}(\alpha_i^-))_\#$ for every $k \leq n$. Moreover, the family $(h^k(\alpha_i^-))_{k \leq n}$ is homotopically disjoint from Δ , according to the third hypothesis of the lemma. Hence $g^k(\alpha_i)$ is disjoint from Δ and $f(\Delta)$ for every $k \leq n$. It follows that $\{h^k(x_i)\}_{k \leq n}$ is included in D . Since the orbits of a Brouwer homeomorphism are proper, this gives a contradiction: D should be disjoint from \mathcal{O} , hence it is a bigone, which is also not possible because Δ and $f(\Delta)$ are in minimal position.

If Δ and $f(\Delta)$ have exactly one intersection point. The set $\Delta \cup f(\Delta)$ splits \mathbb{R}^2 into 4 connected components. Each of those connected components contains at least one orbit of \mathcal{O} : if not, we can find an isotopy relatively to \mathcal{O} which eliminate the intersection point, hence Δ and $f(\Delta)$ are not in minimal position. Choose an orientation on Δ and

consider the induced orientation by f on $f(\Delta)$. Then there exists at least one orbit of \mathcal{O} which is on the left of Δ and on the right of $f(\Delta)$. We claim that this is not possible:

- Δ splits \mathbb{R}^2 into two topological half plane, denoted by \mathcal{P} and \mathcal{Q} .
- Δ splits the orbits of \mathcal{O} into two families: the family P is the orbits included in \mathcal{P} and the family Q is the orbits included in \mathcal{Q} .
- For every orbit \mathcal{O}_i of \mathcal{O} , we have $f(\mathcal{O}_i) = \mathcal{O}_i$.

Then $f(\Delta)$ splits \mathbb{R}^2 into $f(\mathcal{P})$ and $f(\mathcal{Q})$, hence the orbits into $f(P) = P$ and $f(Q) = Q$.

If Δ and $f(\Delta)$ have no intersection point. The set $\Delta \cup f(\Delta)$ splits \mathbb{R}^2 into three connected components. One of them is between Δ and $f(\Delta)$. This component contains at least one orbit, because Δ and $f(\Delta)$ are not isotopic. Now the same argument than in the previous case leads us to a contradiction: choose an orientation on Δ and consider the induced orientation by f on $f(\Delta)$. There exists at least one orbit of \mathcal{O} which is on the left of Δ and on the right of $f(\Delta)$. This is not possible. \square

Corollary 5.4. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Let Z be a stable area for $[h; \mathcal{O}]$ which contains at least two orbits of \mathcal{O} . Choose a complete family of adjacency areas for $[h; \mathcal{O} \cap Z]$. Let Δ be a reducing line for $[h; \mathcal{O} \cap Z]$ which is included in Z and disjoint from every chosen backward adjacency area. Then Δ is a reducing line for $[h; \mathcal{O}]$.*

Proof. Choose a complete hyperbolic metric on $\mathbb{R}^2 - \mathcal{O}$. We suppose that the boundary components of Z and Δ are geodesic. The straightening principle 1.1 gives us $h' \in [h; \mathcal{O}]$ which preserves Z . Let $(\alpha_i^\pm)_i$ be a nice family disjoint from the boundary component of Z , such that every α_i^- of Z is included in an adjacency area of the chosen family.

Since Δ is disjoint from every chosen backward adjacency areas, it is disjoint from α_i^- for every i such that α_i^- is in Z . Hence for every $k \in \mathbb{Z}$, Δ is homotopically disjoint from $h'^k(\alpha_i^-)$ relatively to $\mathcal{O} \cap Z$ (because Δ is isotopic to its image by h' relatively to $\mathcal{O} \cap Z$). Since $h'^k(\alpha_i^-)$ is compact and included in Z , which is preserved by h' , we get that Δ and $h'^k(\alpha_i^-)$ are homotopically disjoint relatively to \mathcal{O} (and not only relatively to $\mathcal{O} \cap Z$).

It follows from lemma 5.3 that Δ is a reducing line for $[h; \mathcal{O}]$. \square

5.2.2 Proof of proposition 2.9

Proposition. 2.9 (Combinatorics of irreducible areas). *Let $[h; \mathcal{O}]$ be a Brouwer mapping class and let Z be an irreducible area for $[h; \mathcal{O}]$. Then:*

1. *The orbits of Z are not all backward adjacent, neither all forward adjacent for $[h; \mathcal{O}]$;*
2. *Z has at least two boundary components;*
3. *The orbits of $[h; \mathcal{O} \cap Z]$ do not alternate.*

Proof of (1). Assume that Z intersects only one forward adjacency area. According to lemma 4.1, every $\alpha_i^- \in Z$ is forward proper. Hence Z is not irreducible.

Proof of (2). Let $[h; \mathcal{O}]$ be a Brouwer mapping class. Let Z be a stable area for $[h; \mathcal{O}]$ which has only one boundary component and at least two orbits. Denote by L this boundary component. Suppose that Z is not a translation area. We will find a reducing line for $[h; \mathcal{O}]$, included in Z and non isotopic to L . Let $(\alpha_i^\pm)_i$ be a nice family for $[h; \mathcal{O}]$ disjoint from L . There is a subfamily of $(\alpha_i^\pm)_i$ which is a nice family for $[h; \mathcal{O} \cap Z]$. Denote by $(\beta_i^\pm)_i$ this subfamily. We consider the cyclic order of $(\beta_i^\pm)_i$, and look where is L in this cyclic order: the position of L in the cyclic order is the position of L in the plane relatively to the half homotopy streamlines generated by the β_i^\pm 's. There are two different cases:

- (a). If L is between two backward proper arcs or between two forward proper arcs in the cyclic order of $(\beta_i^\pm)_i$;
- (b). If L is between one backward proper arc and one forward proper arc in the cyclic order of $(\beta_i^\pm)_i$.

Case (a). If L is between two backward proper arcs or between two forward proper arcs in the cyclic order of $(\beta_i^\pm)_i$, we claim that there exists an adjacency area for $[h; \mathcal{O} \cap Z]$ which contains L . Indeed, assume that L is between two backward proper arcs (the same proof holds with two forward proper arcs, replacing h by h^{-1} when it is necessary). Denote by β_i^- and β_j^- this two backward proper arcs, and suppose their endpoints are respectively $x_i, h(x_i)$ and $x_j, h(x_j)$. Then there exists an arc γ disjoint from L , whose endpoints are $h(x_i)$ and $h(x_j)$ and such that one connected component of the complement of $T^-(\beta_i^-, h, \mathcal{O}) \cup \gamma \cup T^-(\beta_j^-, h, \mathcal{O})$ does not intersect $\mathcal{O} \cap Z$. Now Handel's theorem 3.1 implies that there exists a reducing line Δ for $[h; \mathcal{O} \cap Z]$ which is disjoint from every backward adjacency area. Hence Δ is included in Z , and according to corollary 5.4, it is a reducing line for $[h; \mathcal{O}]$.

Case (b). Assume L is between one backward proper arc and one forward proper arc in the cyclic order of $(\beta_i^\pm)_i$. Denote by β_i^- and β_j^+ this two arcs. Following the construction 3.1, we get a complete family of adjacency areas for $[h; \mathcal{O} \cap Z]$ disjoint from L . Now let Δ be a reducing line for $[h; \mathcal{O} \cap Z]$ given by lemma 3.5, i.e. such that the intersection between Δ and the complement of the forward adjacency areas of $[h; \mathcal{O} \cap Z]$ is bounded. It follows that Δ intersects L at most in a finite set (because L is disjoint from the adjacency areas). Isotopying Δ if this set is not empty, we can suppose that $\Delta \cap L = \emptyset$. Since Δ is also disjoint from every backward adjacency areas of $[h; \mathcal{O} \cap Z]$, according to corollary 5.4, it is a reducing line for $[h; \mathcal{O}]$.

Proof of (3). This is a consequence of lemma 3.6 of [LR13]: every family of alternating orbits satisfies the uniqueness of homotopy translation arcs. Suppose that the orbits of $[h; \mathcal{O} \cap Z]$ are alternate. According to this lemma and to proposition 2.3, if $(\alpha_i^\pm)_i$ is a nice family for $[h; \mathcal{O}]$ disjoint from the boundary components of Z , then for every i such

that α_i^- is in Z , α_i^- and α_i^+ are isotopic relatively to $\mathcal{O} \cap Z$. Since they are included in Z , there are also isotopic relatively to \mathcal{O} . Hence for such an i , $T(\alpha_i^-, h, \mathcal{O})$ is a proper streamline. At least one of the boundary components of a tubular neighborhood of this streamline is a reducing line included in Z . Hence Z is not irreducible. \square

5.3 Corollaries of proposition 2.9

5.3.1 Proof of corollary 2.10

Corollary. 2.10 *Let $[h; \mathcal{O}]$ be a Brouwer mapping class relatively to r orbits. Denote by $2r'$ the number of adjacency subfamilies of $[h; \mathcal{O}]$. If $r' = 1, 2$ or r , then $[h; \mathcal{O}]$ is a flow class.*

Proof. If $[h; \mathcal{O}]$ is not a flow, then it has an irreducible area for some reducing set (according to proposition 2.6). This irreducible area has at least two boundary components (according to proposition 2.9), which are reducing lines. Denote by Δ_1 and Δ_2 those two boundary components. The complement of $\Delta_1 \cup \Delta_2$ has three components, denoted by Z_1 , Z_2 and Z . Assume that Z is the area in the middle, which contains the irreducible area. Choose a nice family $(\alpha_i^\pm)_i$ for $[h; \mathcal{O}]$ which is disjoint from Δ_1 and Δ_2 (use proposition 2.3). Since Z contains an irreducible area, according to proposition 2.9 it intersects at least two different backward adjacency areas of $[h; \mathcal{O}]$ and at least two different forward adjacency areas of $[h; \mathcal{O}]$, and the orbits of $[h; \mathcal{O} \cap Z]$ do not alternate. Hence the situation is the one of figure 27: there exists a subfamily $(\alpha_{i_1}^-, \alpha_{i_2}^+, \alpha_{i_3}^-, \alpha_{i_4}^+)$ of $(\alpha_i^\pm)_i$ containing only arcs included in Z and such that the cyclic order of this subfamily is $(\alpha_{i_1}^-, \alpha_{i_2}^+, \alpha_{i_4}^+, \alpha_{i_3}^-)$, with Δ_1 between $\alpha_{i_2}^+$ and $\alpha_{i_4}^+$ and Δ_2 between $\alpha_{i_3}^-$ and $\alpha_{i_1}^-$. Since Z_1 and Z_2 contain at least one orbit (because Δ_1 and Δ_2 are reducing lines), there exist a backward proper arc of $(\alpha_i^\pm)_i$ in Z_1 , denoted by $\alpha_{i_5}^-$, and a forward proper arc of $(\alpha_i^\pm)_i$ in Z_2 , denoted by $\alpha_{i_6}^+$. It follows that $(\alpha_i^\pm)_i$ has a subfamily of arcs whose cyclic order at infinity is $(\alpha_{i_1}^-, \alpha_{i_2}^+, \alpha_{i_5}^-, \alpha_{i_4}^+, \alpha_{i_3}^-, \alpha_{i_6}^+)$. Hence $r' \geq 3$. It was shown in [LR13], lemma 6.6, that $r' < r$. \square

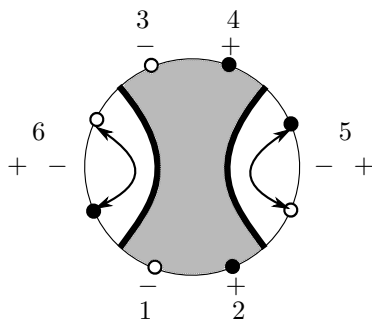


Figure 27: Combinatorics of an irreducible area

5.3.2 Proof of corollary 5.5

Corollary 5.5. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class relatively to r orbits.*

1. *If $r \geq 3$, there exist at least two disjoint and non isotopic reducing lines for $[h; \mathcal{O}]$;*
2. *If $r \geq 2$, there exist at least two translation areas for $[h; \mathcal{O}]$ which have exactly one boundary component;*
3. *There exists a nice family $(\alpha_i^\pm)_i$ and $j \neq k$ such that:*
 - *Relatively to \mathcal{O} , α_j^- is isotopic to α_j^+ and α_k^- is isotopic to α_k^+ ;*
 - *In the cyclic order, α_j^- and α_j^+ (respectively α_k^- and α_k^+) are neighbors.*

Proof of (1). Let r be greater than 2. Theorem 1.3 gives us a first reducing line. This line splits the plane into two stable areas. One of them contains at least two orbits. According to (2) of proposition 2.9, every stable area with one boundary component and which contains at least two orbits contains at least one reducing line non isotopic to the boundary component: this gives a second reducing line. \square

Proof of (2) and (3). If $r = 2$ then $[h; \mathcal{O}]$ any reducing line split the plane into two translation areas. If $r \geq 3$ we find two reducing lines as done in the proof of (1). In the complement of these two reducing lines we have in particular two adjacency areas with one boundary component. In each of these areas, finding again a reducing line in the area as done in (1) if necessary, inductively we find a stable area with one orbit and one boundary components: this gives (2) and (3). \square

6 Deflectors

This section is independent of sections 3, 4 and 5. The main result is proposition 6.1, that we will need to prove theorem 2.13.

Proposition 6.1. *Let $\tau : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the horizontal translation $(x, y) \mapsto (x + 1, y)$. Let $n \in \mathbb{N}$. Let $(\alpha_i)_{1 \leq i \leq n}$ and $(\beta_i)_{1 \leq i \leq n}$ be two families of translation arcs for τ such that:*

- *For every i , α_i and β_i join $(0, i)$ to $(1, i)$.*
- *The α_i 's (respectively the β_i 's) are mutually disjoint.*

Then there exists a homeomorphism μ of \mathbb{R}^2 with a compact support and such that:

1. $\mu(\mathbb{Z} \times \{1, \dots, n\}) = \mathbb{Z} \times \{1, \dots, n\}$;
2. *For every sufficiently large $k \in \mathbb{N}$, for every i , $(\mu\tau)^k(\alpha_i)$ is isotopic relatively to $\mathbb{Z} \times \{1, \dots, n\}$ to $\tau^k(\beta_i)$;*

3. $\mu\tau$ is a Brouwer homeomorphism; More precisely, μ is a finite product of τ -free half twists disjointly supported.

Definition 6.1. Such a homeomorphism is called a *deflector* associated to $(\alpha_i, \beta_i)_{1 \leq i \leq n}$.

Let \mathcal{C}_n be the open vertical cylinder with n marked points at distinct heights. Recall that $MCG(\mathcal{C}_n)$ is defined as the quotient of the group of homeomorphisms of the cylinder, fixing each boundary puncture and fixing the set of marked points (not necessary point wise), by its connected component of the identity (for the compact-open topology). In particular, it is the subgroup of the braid group of the $(n+2)$ -punctured sphere $B_{n+1}(\mathbb{S}^2)$ which fixes two punctures.

We use the following lemma in the proof of proposition 6.1.

Lemma 6.2. *Let $(\gamma_i)_{1 \leq i \leq n}$ be a family of mutually disjoint simple closed curves on \mathcal{C}_n , such that each curve contains exactly one marked point, and such that each curve is isotopic in the cylinder without marked point to the separating circle. Let $(\gamma'_i)_{1 \leq i \leq n}$ be the family of disjoint horizontal circles on the cylinder, such that each γ'_i contains a marked point.*

Then there exists $\phi \in MCG(\mathcal{C}_n)$ such that:

- *For every i , there exists j such that $\phi(\gamma_i)$ is isotopic to γ'_j in \mathcal{C}_n ;*
- *ϕ is a finite product of half-twists.*

Proof of lemma 6.2. We denote again by γ_i, γ'_i the isotopy classes of γ_i, γ'_i when there is no confusion. Let $\varphi \in MCG(\mathcal{C}_n)$ such that $(\varphi(\gamma_i))_i = (\gamma'_i)_i$. Then if T is a product of horizontal Dehn twists (around one of the γ'_i 's or around a boundary component), $T\varphi$ coincides with φ on the γ_i 's.

The group $MCG(\mathcal{C}_n)$ as a quotient of a subgroup of the braid group. Denote by B_n the usual braid group, i.e. the mapping class group of the disk with $n+1$ marked point, denoted by x_1, \dots, x_{n+1} . Let G be the subgroup of B_n which fixes x_{n+1} . Thus $MCG(\mathcal{C}_n)$ is the quotient of G by its center (which is generated by the Dehn twist around the boundary component of the disk).

We will need the three followings to prove the lemma.

(1) Linking number of a pure braid. Denote by P_n the subgroup of B_n composed by the pure braids, i.e. the braids which fix point wise the marked point x_1, \dots, x_n . Let $\rho \in P_n$ be a pure braid. In the geometric representation of ρ , for every $i \leq n$, the strand from x_i can turn around the strand from x_{n+1} clockwise or counterclockwise. We count $+1$ each time it turns around clockwise, and -1 each time it turns around counterclockwise. We call *linking number of x_i around x_{n+1}* and denote by $\epsilon_i(\rho)$ the sum obtained when we look over the strand from x_i . We define the *total linking number of ρ* as the sum $\epsilon(\rho) := \sum_{i=1}^n \epsilon_i(\rho)$. Note that ϵ is a morphism from P_n to \mathbb{Z} .

(2) Special form of a pure braid (see for example [KT08]). The braid $A_{i,j} \in B_n$ is usually defined as in figure 28. More precisely, if we denote by σ_k the usual half twists

which generate B_n , we have:

$$A_{i,j} = \sigma_{i-1}\sigma_{i-2}\dots\sigma_{j+1}\sigma_j^2\sigma_{j+1}^{-1}\dots\sigma_{i-2}^{-1}\sigma_{i-1}^{-1}.$$

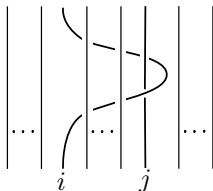


Figure 28: The braid $A_{i,j}$.

Claim A. (see for example [KT08]) Let $\rho \in P_n$ be a pure braid. Then ρ can be written as: $\rho = \beta_2 \dots \beta_n \beta_{n+1}$, where every β_k is in the free group generated by $A_{1,k}, \dots, A_{k-1,k}$.

(3) Claim B. Let ρ be in the group generated by $A_{1,n+1}, \dots, A_{n,n+1}$. Assume that the linking number of ρ is trivial. Then ρ can be written as a finite product of half twists supported in topological disks disjoint from x_{n+1} .

Proof of claim B. Such an element can be written as:

$$\rho = A_{i_1,n+1}^{k_1} A_{i_2,n+1}^{k_2} \dots A_{i_l,n+1}^{k_l} A_{i_{l+1},n+1}^{-(k_1+k_2+\dots+k_l)},$$

where the k_j 's are in \mathbb{Z} , the i_j 's are integers between 1 and n , and l is a non negative integer. Note that $\epsilon(A_{k,n+1}) = 1$ for every k . Since ϵ is a morphism, $\epsilon(\rho) = 0$ implies that the sum of the powers is trivial.

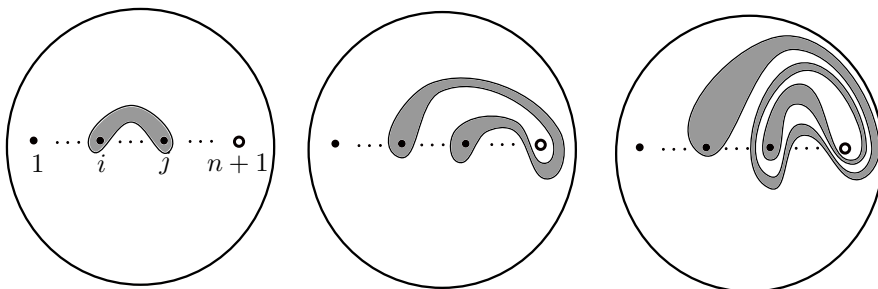


Figure 29: Topological disks $D_{i,j}$, $\sigma_{j,n+1}^2(D_{i,j})$ and $\sigma_{j,n+1}^4(D_{i,j})$.

Denote by $\sigma_{i,j}$ the half twist supported in a topological disk $D_{i,j}$ containing x_i and x_j and lying above every other marked points, as the left item of figure 29. Note that:

- The conjugate of $\sigma_{i,j}$ by $\sigma_{j,n+1}^{2k}$ is the half twist supported in $\sigma_{j,n+1}^{2k}(D_{i,j})$ (see figure 29);

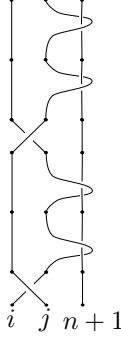


Figure 30: $A_{i,n+1}^2 A_{j,n+1}^{-2} = \sigma_{j,n+1}^4 \sigma_{i,j} \sigma_{j,n+1}^{-4} \sigma_{i,j}^{-1}$.

- $A_{i,n+1}^k A_{j,n+1}^{-k} = \sigma_{j,n+1}^{2k} \sigma_{i,j} \sigma_{j,n+1}^{-2k} \sigma_{i,j}^{-1}$ (see figure 30).

Thus we know how to realize every $A_{i,n+1}^k A_{j,n+1}^{-k}$ as a product of 4 half twists supported in $D_{i,j}$ and $\sigma_{j,n+1}^{2k}(D_{i,j})$, hence as a product of half twists supported in topological disks disjoint from x_{n+1} . Now remark that ρ can be written as:

$$\rho = A_{i_1,n+1}^{k_1} A_{i_2,n+1}^{-k_1} A_{i_2,n+1}^{k_1+k_2} A_{i_3,n+1}^{-(k_1+k_2)} \dots A_{i_l,n+1}^{k_1+\dots+k_l} A_{i_{l+1},n+1}^{-(k_1+k_2+\dots+k_l)}.$$

Hence we know how ρ as a finite product of half twists supported in topological disks disjoint from x_{n+1} . \square

Back to the proof of lemma 6.2. Choose a topological disk K of \mathcal{C} which contains every x_i for $i \leq n$. Every element of B_n supported in K is in G . Choose a lift $\hat{\varphi}$ of φ in G . Compose $\hat{\varphi}$ with an element $\sigma \in G$ supported in K and such that $\sigma\hat{\varphi}$ is a pure braid.

(1) Note that the linking number of $\sigma\hat{\varphi}$ does not depend on σ , because its supported in K . Up to composing φ by a finite product of horizontal Dehn twists, we can assume that $\epsilon(\sigma\hat{\varphi}) = 0$.

(2) Since $\sigma\hat{\varphi}$ is a pure braid, according to claim A, we can write it as:

$$\sigma\hat{\varphi} = \beta_2 \dots \beta_n \beta_{n+1},$$

where every β_k is in the free group generated by $A_{1,k}, \dots, A_{k-1,k}$.

For every $i \leq k \leq n$, $A_{i,k}$ is supported in K , hence $\beta_n \dots \beta_2$ is supported in K , hence $\epsilon(\beta_k) = 0$ for every $k \leq n$. Since ϵ is a morphism, we have also $\epsilon(\beta_{n+1}) = 0$.

(3) Because the braid group B_{n-1} is generated by the usual half twists (see for example [KT08]), we know how to write any element supported in K as a product of half twists supported in topological disks disjoint from x_{n+1} . Now applying claim B to β_{n+1} , we also know how to write β_{n+1} as a product of half twists supported in topological disks disjoint from x_{n+1} . Thus we know how to write $\sigma\hat{\varphi}$, hence $\hat{\varphi}$, as a finite product of half twists supported in topological disks disjoint from x_{n+1} . This product defines an element of $MCG(\mathcal{C}_n)$ which satisfies the property of the lemma. \square

Proof of proposition 6.1. Denote by \mathcal{C} the vertical cylinder (quotient of \mathbb{R}^2 by τ) and by \mathcal{C}_n the cylinder \mathcal{C} with n marked points (quotient of $\mathbb{R}^2 - \mathbb{Z} \times \{1, \dots, n\}$ by τ). Let π be the quotient map. For every i , we denote by $\hat{\alpha}_i$, respectively $\hat{\beta}_i$, the isotopy class of $\pi(\alpha_i)$, respectively $\pi(\beta_i)$, in \mathcal{C}_n . There exists $\psi \in MCG(\mathcal{C}_n)$ such that $\psi(\hat{\alpha}_i)$ is isotopic in \mathcal{C}_n to the horizontal circle containing the marked point $x_i = \pi(\mathbb{Z} \times \{i\})$. There exists $\chi \in MCG(\mathcal{C}_n)$ such that $\chi(\hat{\beta}_i)$ is isotopic in \mathcal{C}_n to the horizontal circle containing the marked point $x_i = \pi(\mathbb{Z} \times \{i\})$. We set $\phi := \chi^{-1}\psi$. Hence $\phi(\hat{\alpha}_i)$ is isotopic in \mathcal{C}_n to $\hat{\beta}_i$. According to lemma 6.2, we can assume that $\phi := \nu_1 \dots \nu_k$ is a finite product of half twists supported in topological disks of \mathcal{C} . We want to use ϕ to construct the desired homeomorphism μ (figure 31 gives the main idea of the proof).

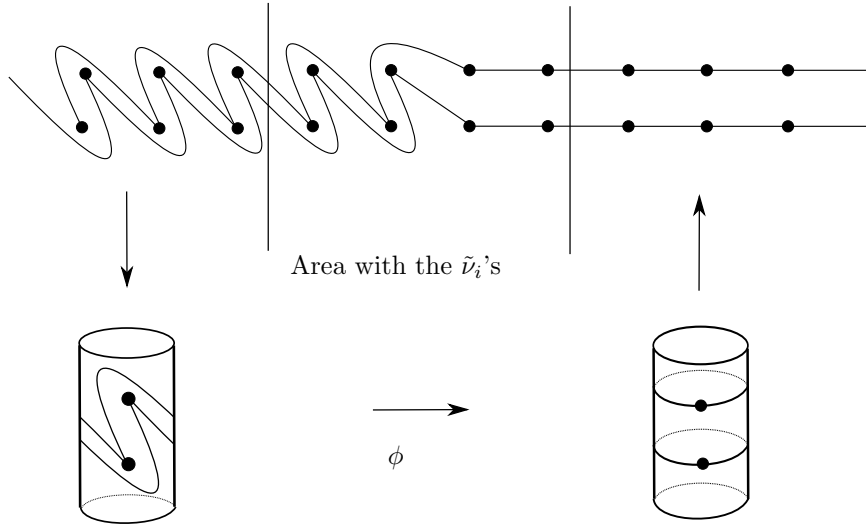


Figure 31: Half streamlines $T^+(\alpha_i, \mu\tau, \mathbb{Z} \times \{1, \dots, n\})$.

Local lift of a half twist. We first construct a homeomorphism of \mathbb{R}^2 which lift the action of *one* half twist of \mathcal{C}_n to the lift of the curves. Let ν be a half twist of \mathcal{C}_n . Choose a lift D_ν of the support of ν in \mathbb{R}^2 . Denote by $\tilde{\nu}$ the homeomorphism of \mathbb{R}^2 such that $\pi\tilde{\nu}|_{D_\nu} := \nu\pi|_{D_\nu}$, and such that $\tilde{\nu}$ coincides with Id outside D_ν . Note that $\mathbb{Z} \times \{1, \dots, k\}$ is preserved by $\tilde{\nu}\tau$. We say that $\tilde{\nu}$ is a *local lift* of ν .

Choice of disks. To lift the action of $\phi := \nu_1 \dots \nu_k$, we choose a local lift $\tilde{\nu}_k$ of ν_k supported in a disk D_k on the right of the α_i 's and for every $1 \leq j < k$, we choose a local lift $\tilde{\nu}_j$ of ν_j supported in a disk D_j on the right of D_{j+1} (and disjoint from D_{j+1}).

Conclusion. Let $\tilde{\mu}$ be $\tilde{\nu}_1 \dots \tilde{\nu}_k$. For every x which is on the left of the D_j 's, for every n sufficiently large, there exists $(n_j)_j \in \mathbb{N}^{k+1}$ such that:

$$(\tilde{\mu}\tau)^n = \tau^{n_{k+1}}\tilde{\nu}_1 \dots \tilde{\nu}_{k-1}\tau^{n_2}\tilde{\nu}_k\tau^{n_1}(x).$$

Moreover, we have the followings:

$$\pi\tilde{\nu} = \nu\pi;$$

$$\pi\tau = \pi.$$

Hence we get:

$$\pi(\tilde{\mu}\tau)^n\alpha_i = \pi\beta_i.$$

□

7 Classification relatively to 4 orbits

7.1 Identification of the determinant diagrams

Here we want to identify which diagrams are determinant diagrams. In the next section, we will study the diagrams which are not determinant. Note that all the diagrams with four orbits are represented in the appendix A.

For every Brouwer mapping class relatively to 4 orbits, we denote by $2r'$ the number of sub-families of adjacency.

Proposition 7.1. *A diagram for a Brouwer mapping class relatively to 4 orbits is non determinant if and only if it is one of the seven of figure 32.*

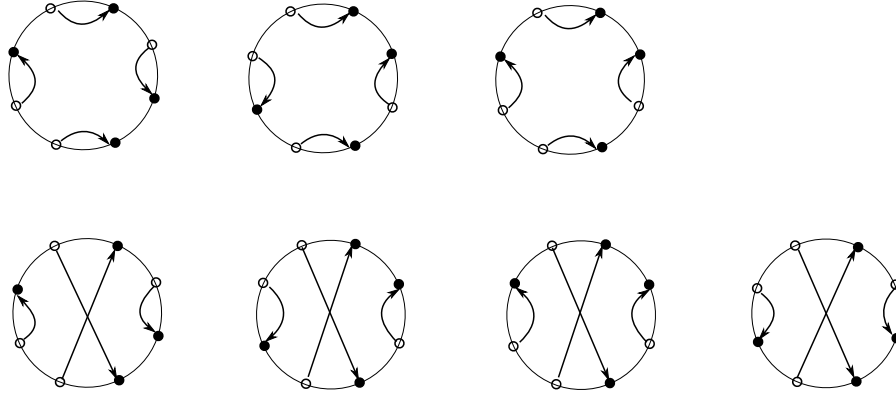


Figure 32: Non determinant diagrams for $r = 4$

Proof. According to proposition 2.9, every irreducible area for Brouwer mapping classes relatively to 4 orbits is as in figure 33. □

7.2 Study of the non determinant diagrams

7.2.1 Brouwer mapping classes which realize non determinant diagrams

If h is a homeomorphism of the plane, recall that an h -free disk is a topological disk D which is disjoint from every $h^n(D)$, with $n \neq 0$. If $[h; \mathcal{O}]$ is a Brouwer mapping class and

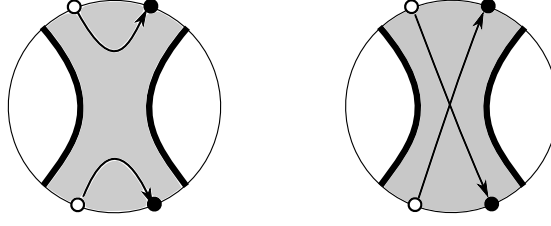


Figure 33: Possible irreducible areas for $r = 4$

if D is an h -free disk containing exactly two points of \mathcal{O} , then we call *free half twist* any half twist supported in D and permuting the two points of $D \cap \mathcal{O}$.

Remark 7.1. Each non determinant diagram can be realized by a Brouwer mapping class:

- For each non determinant diagram without crossing, there exists a flow having this diagram relatively to some of its orbits (see lemma 1.7 of [LR13]);
- For each non determinant diagram with crossing, we can obtain it by composing a flow by a free half twist (as in example B of section 1).

7.2.2 Tangle of the irreducible area

Let $[h; \mathcal{O}]$ be a Brouwer mapping class relatively to 4 orbits which is not a flow class: the diagram with walls of $[h; \mathcal{O}]$ is as in figure 34. To simplify the notations, suppose that the two orbits of the irreducible areas are \mathcal{O}_1 and \mathcal{O}_2 , with α_1^\pm and α_2^\pm 34. To define the tangle, we will forget about \mathcal{O}_3 and \mathcal{O}_4 for a while.

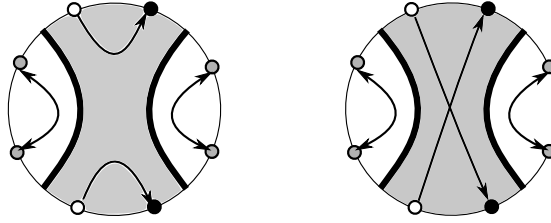


Figure 34: Diagrams with an irreducible area for $r = 4$.

Choose a nice family $(\alpha_i^\pm)_i$ for $[h; \mathcal{O}]$. Choose a complete hyperbolic metric on $\mathbb{R}^2 - (\mathcal{O}_1 \cup \mathcal{O}_2)$. According to corollary 2.10 and proposition 2.7 (see also Handel [Han99]), the diagram relatively to 2 orbits is a total conjugacy invariant, hence $[h; \mathcal{O}_1 \cup \mathcal{O}_2]$ is a translation class. It follows that the 4 homotopic trajectories relative to $\mathcal{O}_1 \cup \mathcal{O}_2$:

$$T_1^+ := \bigcup_{k \in \mathbb{Z}} h^k(\alpha_1^+)_{\#}$$

$$\begin{aligned}
T_1^- &:= \bigcup_{k \in \mathbb{Z}} h^k(\alpha_1^-)_\# \\
T_2^+ &:= \bigcup_{k \in \mathbb{Z}} h^k(\alpha_2^+)_\# \\
T_2^- &:= \bigcup_{k \in \mathbb{Z}} h^k(\alpha_2^-)_\#
\end{aligned}$$

are proper homotopic lines. Moreover, the T_i^+ 's (respectively the T_i^- 's) are mutually disjoint. Let ϕ be a homeomorphism of the plane that preserves the orientation and sends, for $i = 1, 2$:

- T_i^- on $\mathbb{R} \times \{i\}$ and T_2^- on $\mathbb{R} \times \{2\}$;
- $\{x_i\}$ on $(0, i)$ and $\{h(x_i)\}$ on $(1, i)$, where x_i and $h(x_i)$ are the endpoints of α_i^- ;
- \mathcal{O}_i on $\mathbb{Z} \times \{i\}$.

Let τ be the horizontal translation of the plane which maps $(x, y) \in \mathbb{R}^2$ to $(x + 1, y)$.

Let π be the quotient map which quotients $\mathbb{R}^2 - (\mathbb{Z} \times \{1, 2\})$ by τ and let \mathcal{C}_2 denotes the quotient pointed cylinder. Note that if we consider \mathcal{C}_2 as a vertical cylinder, $\pi(\phi(\alpha_i^-))$ is homotopic in \mathcal{C}_2 to a horizontal circle for $i = 1, 2$ (see figure 35 for an example).

Lemma 7.2. *With the previous notations, the homotopy classes of the arcs $\pi(\phi(\alpha_i^+))$ in \mathcal{C}_2 are independent of ϕ .*

Proof. If ψ is another homeomorphism with the same properties, then ϕ and ψ coincide on the two topological lines T_1^- and T_2^- . According to lemma 5.1, ϕ and ψ are isotopic relatively to $\mathcal{O}_1 \cup \mathcal{O}_2$, hence $\phi(\alpha_i^+)$ is isotopic to $\psi(\alpha_i^+)$ for $i = 1, 2$. \square

Denote by γ a curve which is disjoint from $\pi(\phi(\alpha_1^+))$ and $\pi(\phi(\alpha_2^+))$ and which separates \mathcal{C}_2 into two cylinders with puncture, each of them containing one of the $\pi(\phi(\alpha_i^+))$'s. Note that γ is unique up to isotopy in \mathcal{C}_2 .

Definition 7.1. We say that the isotopy class of $\gamma \in \mathcal{C}_2$ is the *tangle of the irreducible area of $[h; \mathcal{O}]$ relative to $(\alpha_i^\pm)_i$* .

Remark 7.2. Note that γ is never a horizontal circle. Indeed, we could get a horizontal curve only if we had α_i^- isotopic to α_i^+ relatively to $\mathcal{O}_1 \cup \mathcal{O}_2$ for $i = 1, 2$. In this situation we get proper streamlines for every orbit, hence $[h; \mathcal{O}]$ is a flow class: this gives a contradiction because we assumed that there exists an irreducible area for $[h; \mathcal{O}]$.

This relative tangle depend on the choice of the nice family $(\alpha_i^\pm)_i$, hence it is not a conjugacy invariant. However, we have the following lemma:

Lemma 7.3. *Let $[h; \mathcal{O}]$ be a Brouwer mapping class relatively to 4 orbits. Suppose that $[h; \mathcal{O}]$ is not a flow class. Then if $(\alpha_i^\pm)_i$ and $(\beta_i^\pm)_i$ are two nice families for $[h; \mathcal{O}]$ disjoint from the walls, then for every i there exists n_i such that α_i^- , respectively α_i^+ , is isotopic to $h^{n_i}(\beta_i^-)$, respectively $h^{n_i}(\beta_i^+)$ relatively to \mathcal{O} .*

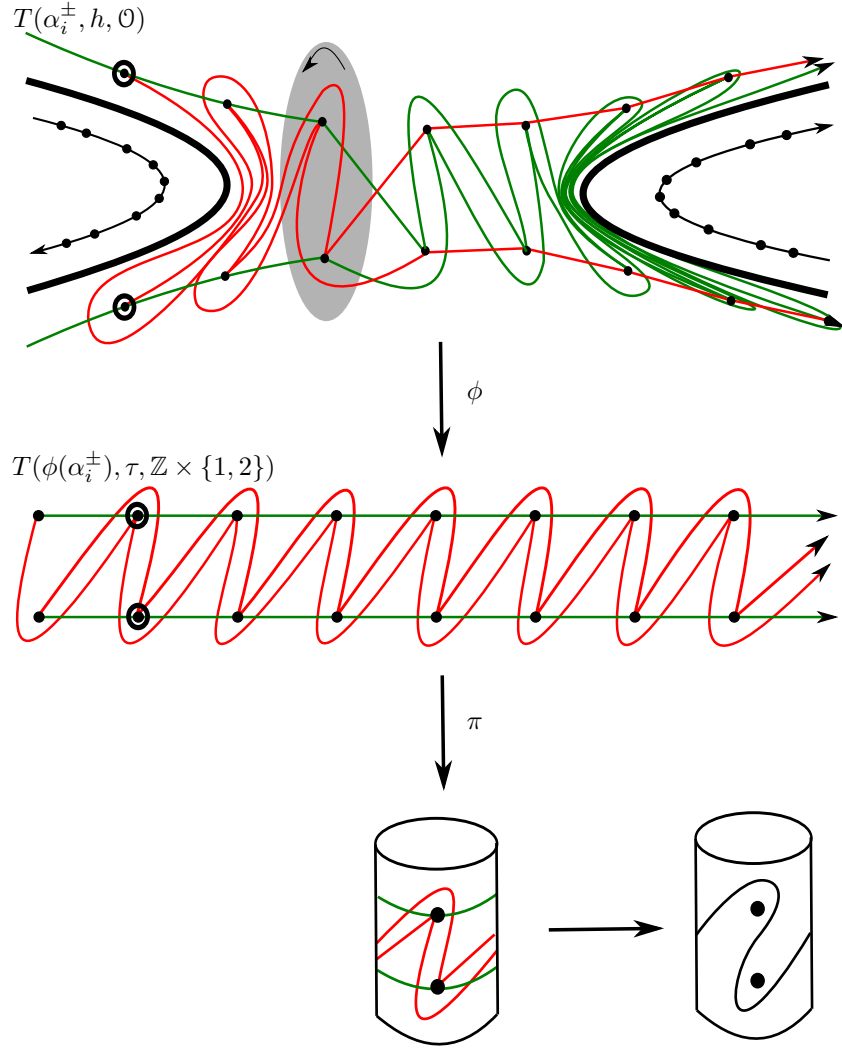


Figure 35: Definition of the relative tangent for the example B of section 1 (Brouwer class of the product of a free half-twist with a flow).

Proof. This will follow from the description of the adjacency areas of $[h; \mathcal{O}]$. Choose a complete family of adjacency areas for $[h; \mathcal{O}]$ and a representative $\{\Delta_1, \Delta_2\}$ of the set of walls. For every nice family $(\alpha_i^\pm)_i$ disjoint from $\Delta_1 \cup \Delta_2$, there exists $(m_i, n_i)_i \in (\mathbb{Z}^2)^4$ such that $h_i^m(\alpha_i^-)$ and $h_i^n(\alpha_i^+)$ are included in adjacency areas. If we fix the endpoints in the adjacency area, there is only one isotopy class of homotopic translation arc included in the chosen adjacency area and disjoint from Δ_1 and Δ_2 : indeed there is only one isotopy class of translation arcs for Brouwer class relatively to one orbit (according to corollary 6.3 of Handel [Han99]). \square

We denote by T the left Dehn twist around a separating horizontal circle between

the two punctures in \mathbb{C}_2 .

Lemma 7.4. *With the previous notations, if $\gamma \in \mathbb{C}_2$ (respectively $\gamma' \in \mathbb{C}_2$) is the tangle of the irreducible area of $[h; \mathcal{O}]$ relative to $(\alpha_i^\pm)_i$ (respectively $(\beta_i^\pm)_i$), then there exists $n \in \mathbb{Z}$ such that $\gamma = T^n \gamma'$.*

Proof. This is a consequence of lemma 7.3: this lemma implies that if ϕ is as in the previous notations, for $i = 1, 2$ we have:

$$T(\phi(\alpha_i^\pm), \tau, \mathbb{Z} \times \{1, 2\}) = T(\phi(h^{n_i}(\beta_i^\pm)), \tau, \mathbb{Z} \times \{1, 2\}) = T(\phi(\beta_i^\pm), \tau, \mathbb{Z} \times \{1, 2\}).$$

Moreover, we have $\phi(h^{n_i}(x_i)) = \tau^{n_i}(\phi(x_i))$.

Since $\phi(x_i) = (0, i)$, it follows that $\phi(h^{n_i}(x_i)) = (n_i, i)$, hence:

$$\pi(\phi(\beta_i^\pm)) = T^{n_1 - n_2}(\pi(\phi(\alpha_i^\pm))).$$

□

Definition 7.2. With the previous notations, we define the *tangle* of $[h; \mathcal{O}]$ to be the isotopy class $\gamma \in \mathbb{C}_2$ up to composition by T .

By convention, we set that every flow mapping class has trivial tangle.

Corollary 7.5. *The couple (Diagram with walls, Tangle) is a conjugacy invariant for Brouwer mapping classes relatively to 4 orbits.*

We will need the following result in the section 7.3.

Lemma 7.6. *With the previous notations, let γ be the tangle of the irreducible area of $[h; \mathcal{O}]$ relative to $(\alpha_i^\pm)_i$, and let γ' be $T^n(\gamma)$ for some $n \in \mathbb{Z}$, where T is the left Dehn twist as above. Then there exists a nice family $(\beta_i^\pm)_i$ for $[h; \mathcal{O}]$ such that the tangle of the irreducible area of $[h; \mathcal{O}]$ relative to $(\beta_i^\pm)_i$ is the isotopy class of γ' .*

Proof. Define $(\beta_i^\pm)_i$ as $\beta_1^\pm := h^n(\alpha_1^\pm)$ and $\beta_i^\pm := \alpha_i^\pm$ for $i = 2, 3, 4$. □

7.2.3 Realized couples (diagram with walls, tangle)

In section 7.3, we will show that the couple (Diagram with walls, Tangle) is a total conjugacy invariant. Here we find which couples are realized by Brouwer mapping classes relatively to four orbits.

Necessary condition to be realized. Not every couple (Diagram with walls, Tangle) can be realized by a Brouwer mapping class. Indeed, some tangles are associated to non determinant diagram with crossing arrows, and some other tangles are associated to non determinant diagram without crossing arrows. To be more precise, denote by p and q the two marked points of \mathbb{C}_2 , and suppose that p is above q . If γ is a curve of \mathbb{C}_2 representing the tangle, the marked point of \mathbb{C}_2 which is *above* γ (i.e. in the connected component

of the complementary of γ which contains the top of the cylinder \mathcal{C}_2) can be p or q , depending on the tangles.

Moreover, this point represents the orbit whose forward half streamline is above on the picture, hence whose arrow ends above the other on the diagram. It follows that if this point is p , then the diagram is without crossing, and if this point is q , the diagram has a crossing. We say that such a tangle *adapted* to the diagram. See figure 36 for two examples:

- On the tangle of the left, p is above γ , hence it is the tangle of a diagram without crossing;
- On the tangle of the right, q is above γ , hence it is the tangle of a diagram with crossing.

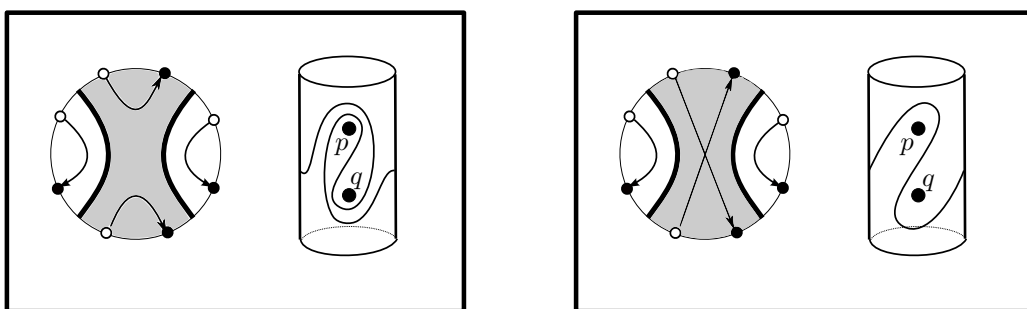


Figure 36: Examples of two diagrams with adapted tangles.

Note that there are infinitely many tangles adapted to each diagram.

Realizing the adapted tangles. Given a couple (diagram, tangle) such that the tangle is adapted to the diagram, we can produce a Brouwer homeomorphism which realizes this couple as follow. Denote by (D, τ) the given couple.

1. (a) If the diagram D does not have crossing arrows, then we define $D' = D$;
(b) If the diagram D has crossing arrows, we consider the diagram D' obtained by exchanging the ends of the two crossing arrows. This is a diagram without crossing arrows;
2. We choose a flow f which realizes the diagram D' without walls and such that there exists a f -free disk which contains one point of each orbit \mathcal{O}_1 and \mathcal{O}_2 (as in example A of section 1), where \mathcal{O}_1 and \mathcal{O}_2 are the two orbits of the irreducible area of D ;
3. By reversing the process of the definition of the tangle given in section 7.2.2, we get two families of translation arcs. Proposition 6.1 provides us a finite product μ of mutually disjointly supported f -free half twists such that the tangle of $[\mu f; \mathcal{O}]$ is τ . Note also that the diagram associated to $[\mu f; \mathcal{O}]$ is D .

7.2.4 Infinitely many Brouwer mapping classes relatively to four orbits

Proposition 7.7. *Up to conjugacy, there are (countably) infinitely many Brouwer mapping classes relatively to 4 orbits.*

Proof. There are infinitely many tangles adapted to each non determinant diagram (see figure 36 for examples). Each of them is realized by a product of a flow with finitely many free half twist disjointly supported (using proposition 6.1, see the second paragraph of section 7.2.3). It follows from corollary 7.5 that there are infinitely many Brouwer mapping classes relatively to 4 orbits. \square

7.3 A total conjugacy invariant

In this section we want to show theorem 2.13, namely that two Brouwer mapping classes relatively to four orbits are conjugated if and only if they have the same invariant couple:

(Diagram with walls, Tangle).

The following lemma 7.8 together with lemma 7.6 give the proof of theorem 2.13: indeed if $[h; \mathcal{O}]$ and $[h'; \mathcal{O}']$ have the same invariant couple, then lemma 7.6 provides us a nice family $(\alpha_i^\pm)_i$ for $[h; \mathcal{O}]$ and a nice family $(\beta_i^\pm)_i$ for $[h'; \mathcal{O}']$ such that $[h; \mathcal{O}]$ and $[h'; \mathcal{O}']$ have the same tangle relative to their nice family. Hence they satisfy the hypothesis of lemma 7.8, which says that they are conjugated.

Lemma 7.8. *Let $[h; \mathcal{O}]$ and $[h'; \mathcal{O}']$ be two Brouwer mapping classes relatively to 4 orbits such that:*

- *They have the same diagram with walls;*
- *There exist two nice families $(\alpha_i^\pm)_i$ for $[h; \mathcal{O}]$ and $(\beta_i^\pm)_i$ for $[h'; \mathcal{O}']$ such that the two Brouwer mapping classes have the same relative tangle relative to their nice family;*

Then $[h; \mathcal{O}]$ and $[h'; \mathcal{O}']$ are conjugated.

Proof. Note that if the diagram with walls of $[h; \mathcal{O}]$ has crossing arrows, then this two crossing arrows are in an irreducible area: indeed they are in the same connected component of the walls, which is not a translation area, hence it is an irreducible area (according to theorem 2.5). If the diagram with walls is without any irreducible area, then $[h; \mathcal{O}]$ and $[h'; \mathcal{O}']$ are conjugated (this is proposition 2.8).

Let us consider the case when there exists an irreducible area. We suppose that the orbits which intersect this area are indexed by 1 and 2. Denote by Z and Z' the irreducible areas of $[h; \mathcal{O}]$ and $[h'; \mathcal{O}']$ respectively. We assume that h preserves Z and h' preserves Z' . Denote by τ the translation of the plane which maps every $(x, y) \in \mathbb{R}^2$ to $(x + 1, y)$. Denote by ϕ a homeomorphism of the plane as needed to define the tangle for $[h; \mathcal{O}]$, i.e. which maps:

- $T_1^- \cup T_2^-$ on $\mathbb{R} \times \{1, 2\}$;

- $\{x_1, x_2\}$ on $\{0\} \times \{1, 2\}$, where x_i and $h(x_i)$ are the endpoints of α_i^- ;
- $\mathcal{O}_1 \cup \mathcal{O}_2$ on $\mathbb{Z} \times \{1, 2\}$;

where $T_i^\pm := \bigcup_{k \in \mathbb{Z}} h^k(\alpha_i^\pm)_\#$.

Likewise, denote by ψ a homeomorphism which maps:

- $T_1'^- \cup T_2'^-$ on $\mathbb{R} \times \{1, 2\}$;
- $\{x'_1, x'_2\}$ on $\{0\} \times \{1, 2\}$, where x'_i and $h(x'_i)$ are the endpoints of β_i^- ;
- $\mathcal{O}'_1 \cup \mathcal{O}'_2$ on $\mathbb{Z} \times \{1, 2\}$;

where $T_i'^\pm := \bigcup_{k \in \mathbb{Z}} h'^k(\alpha_i'^\pm)_\#$.

Since the two classes have the same tangle relatively to their nice families, by definition of ϕ and ψ we have: $(\phi\alpha_i^\pm)_\# = (\psi\beta_i^\pm)_\#$ for $i = 1, 2$. Denote by γ_i^\pm these arcs.

Claim 1. There exists $\phi' \in [\phi; \mathcal{O}_1 \cup \mathcal{O}_2]$ and $\psi' \in [\psi'; \mathcal{O}'_1 \cup \mathcal{O}'_2]$ such that $\phi'Z = \psi'Z'$.

Proof of the claim. Denote by Δ_1 , respectively Δ_2 , the boundary component of Z which is disjoint from $T_1^- \cup T_2^-$, respectively disjoint from $T_1^+ \cup T_2^+$. Since they are disjoint, we may assume that $\phi(\Delta_1)$ is included in the left half plane and $\phi(\Delta_2)$ is included in the right half plane. Similarly, we denote by Δ'_1 and Δ'_2 the boundary components of Z' , disjoint respectively from $T_1'^- \cup T_2'^-$ and $T_1'^+ \cup T_2'^+$, and we assume that $\psi(\Delta'_1)$ is included in the left half plane and $\psi(\Delta'_2)$ is included in the right half plane. Now $\phi(\Delta_1)$ and $\psi(\Delta'_1)$ are lines included in the half left strip between $\mathbb{R} \times \{1\}$ and $\mathbb{R} \times \{2\}$: there exists a homeomorphism λ_1 supported in this half strip and which sends $\phi(\Delta_1)$ on $\psi(\Delta'_1)$. Similarly there exists a homeomorphism λ_2 supported in the right half strip between $\bigcup_{n \geq 0} \tau^n(\gamma_1)$ and $\bigcup_{n \geq 0} \tau^n(\gamma_2)$ and sending $\phi(\Delta_2)$ on $\psi(\Delta'_2)$. It follows that $\lambda_1\lambda_2\phi(Z) = \psi(Z')$. Since $\lambda_1\lambda_2$ is isotopic to the identity relatively to $\mathbb{R} \times \{1, 2\}$, $\lambda_1\lambda_2\phi$ is isotopic to ϕ . \square

Back to the proof of lemma 7.8. Up to isotopying ϕ relatively to $\mathcal{O}_1 \cup \mathcal{O}_2$ and ψ relatively to $\mathcal{O}'_1 \cup \mathcal{O}'_2$ as in claim 1, we may assume that $\phi Z = \psi Z'$.

According to proposition 6.1, there exists a finite product of τ -free half twists disjointly supported and such that for every sufficiently large $k \in \mathbb{N}$, $(\mu\tau)^k(\gamma_i^-)$ is isotopic relatively to $\mathbb{Z} \times \{1, 2\}$ to $\tau^k(\gamma_i^+)$ for $i = 1, 2$. Since μ is compactly supported, we can suppose that this support is included in $\phi(Z) = \psi(Z')$.

Claim 2. With the previous notations, we claim that $[\phi^{-1}\mu\phi h; \mathcal{O}]$ and $[\psi^{-1}\mu\psi h'; \mathcal{O}']$ are flow classes.

Proof of the claim. We do the proof for $[\phi^{-1}\mu\phi h; \mathcal{O}]$: relatively to $\mathcal{O}_1 \cup \mathcal{O}_2$, for every sufficiently large $k \in \mathbb{N}$, for $i = 1, 2$, $(\phi^{-1}\mu\phi h)^k(\alpha_i^-)$ is isotopic to $h^k(\alpha_i^+)$. Because μ is supported in $\phi(Z)$, $\phi^{-1}\mu\phi$ is supported in Z , and since h preserves Z , it follows that $\phi^{-1}\mu\phi h$ also preserves Z . Since α_i^- is included in Z , and since \mathcal{O}_3 and \mathcal{O}_4 do not

intersect Z , $(\phi^{-1}\mu\phi h)^k(\alpha_i^-)$ is isotopic to $h^k(\alpha_i^+)$ relatively to \mathcal{O} (and not only relatively to $\mathcal{O}_1 \cup \mathcal{O}_2$). Since α_i^+ is a forward proper arc for $[h; \mathcal{O}]$, it follows that $T(\alpha_i^-, \phi^{-1}\mu\phi h, \mathcal{O})$ is a proper streamline. Since $\phi^{-1}\mu\phi$ is supported in Z , h is equal to $\phi^{-1}\mu\phi h$ outside Z , hence for $j = 3, 4$, $T(\alpha_j^-, \phi^{-1}\mu\phi h, \mathcal{O}) = T(\alpha_j^-, h, \mathcal{O})$, thus is also a proper streamline. By lemma 5.2, it follows that $[\phi^{-1}\mu\phi h; \mathcal{O}]$ is a flow class. \square

Back to the proof of lemma 7.8. Denote by f and g two flows such that:

$$[f; \mathcal{O}] = [\phi^{-1}\mu\phi h; \mathcal{O}] \text{ and } [g; \mathcal{O}'] = [\psi^{-1}\mu\psi h'; \mathcal{O}'].$$

For every i , denote by T_i , respectively by T'_i , the proper streamline $T(\alpha_i^-, f, \mathcal{O})$, respectively $T(\beta_i^-, g, \mathcal{O}')$. Changing ψ in the complement of Z' if necessary, we assume that $\phi^{-1}\psi$ maps T'_3 on T_3 , \mathcal{O}'_3 on \mathcal{O}_3 , T'_4 on T_4 and \mathcal{O}'_4 on \mathcal{O}_4 . This is possible because the diagrams of $[h; \mathcal{O}]$ and $[h'; \mathcal{O}']$ are the same. Note that for every $k \in \mathbb{Z}$, $\phi^{-1}\psi(\psi^{-1}\mu\psi h')^k(\beta)$ is isotopic to $(\phi^{-1}\mu\phi h)^k(\alpha_i^-)$ relatively to \mathcal{O} for $i = 1, 2$. Hence $T_i = \phi^{-1}\psi(T'_i)$ for every i . According to lemma 5.1, we get:

$$[\phi^{-1}\psi g \psi^{-1}\phi; \mathcal{O}] = [f; \mathcal{O}].$$

Composing both parts by $\phi^{-1}\mu^{-1}\phi$, we can check that:

$$[(\phi^{-1}\psi)h'(\phi^{-1}\psi)^{-1}; \mathcal{O}] = [h; \mathcal{O}].$$

Hence $[h; \mathcal{O}]$ and $[h'; \mathcal{O}']$ are conjugated. \square

A Diagrams with four orbits

Here we represent all the diagrams with four orbits (figures 37, 38, 39 and 40). If a diagram can be obtained with a Brouwer mapping class, then we also draw the possible sets of walls, and color in grey the eventual irreducible areas. We put together in the same dashed box the diagrams which are the same without walls but which have different possible sets of walls. We get three different types of diagrams:

1. The full-framed diagrams are the ones with a Handel's cycle: according to Handel's fixed point theorem (theorem 2.3 of [Han99]), they cannot be obtained with Brouwer homeomorphisms. Also we forget them to describe Brouwer mapping classes relatively to four orbits;
2. The diagrams without irreducible areas are the determinant ones. Every of them can be realized by a Brouwer mapping class (according to lemma 1.7 of [LR13]). Moreover, up to conjugation, this Brouwer mapping class is unique and it is a flow class (propositions 2.6 and 2.7);
3. The diagrams with an irreducible area (in grey) are the eight non determinant ones. Up to conjugation, every of them can be realized by infinitely many Brouwer mapping classes. For those diagrams, the tangle allows us to differentiate the different Brouwer mapping classes (see sections 7.2 and 7.3).

We still denote by $2r'$ the number of families of adjacency.

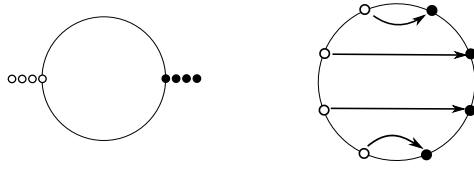


Figure 37: Diagram for $r = 4$ and $r' = 1$.

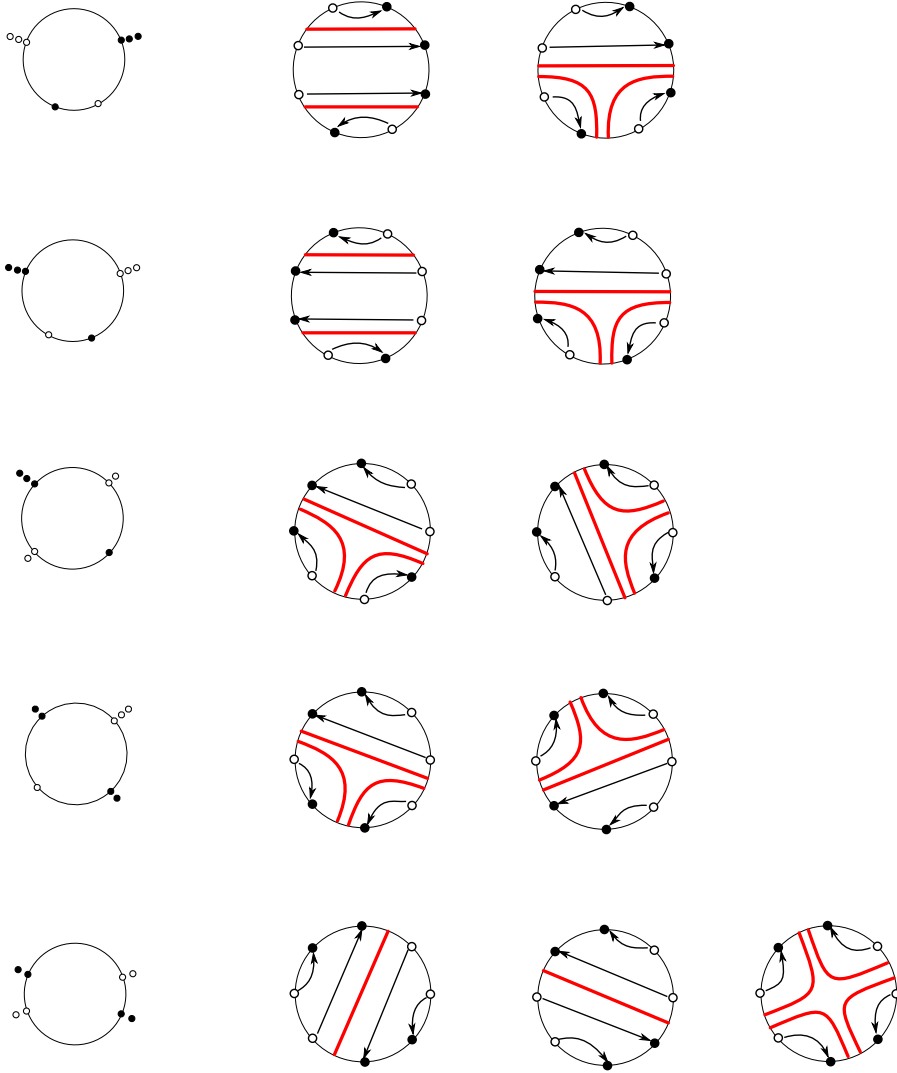


Figure 38: Diagram for $r = 4$ and $r' = 2$.

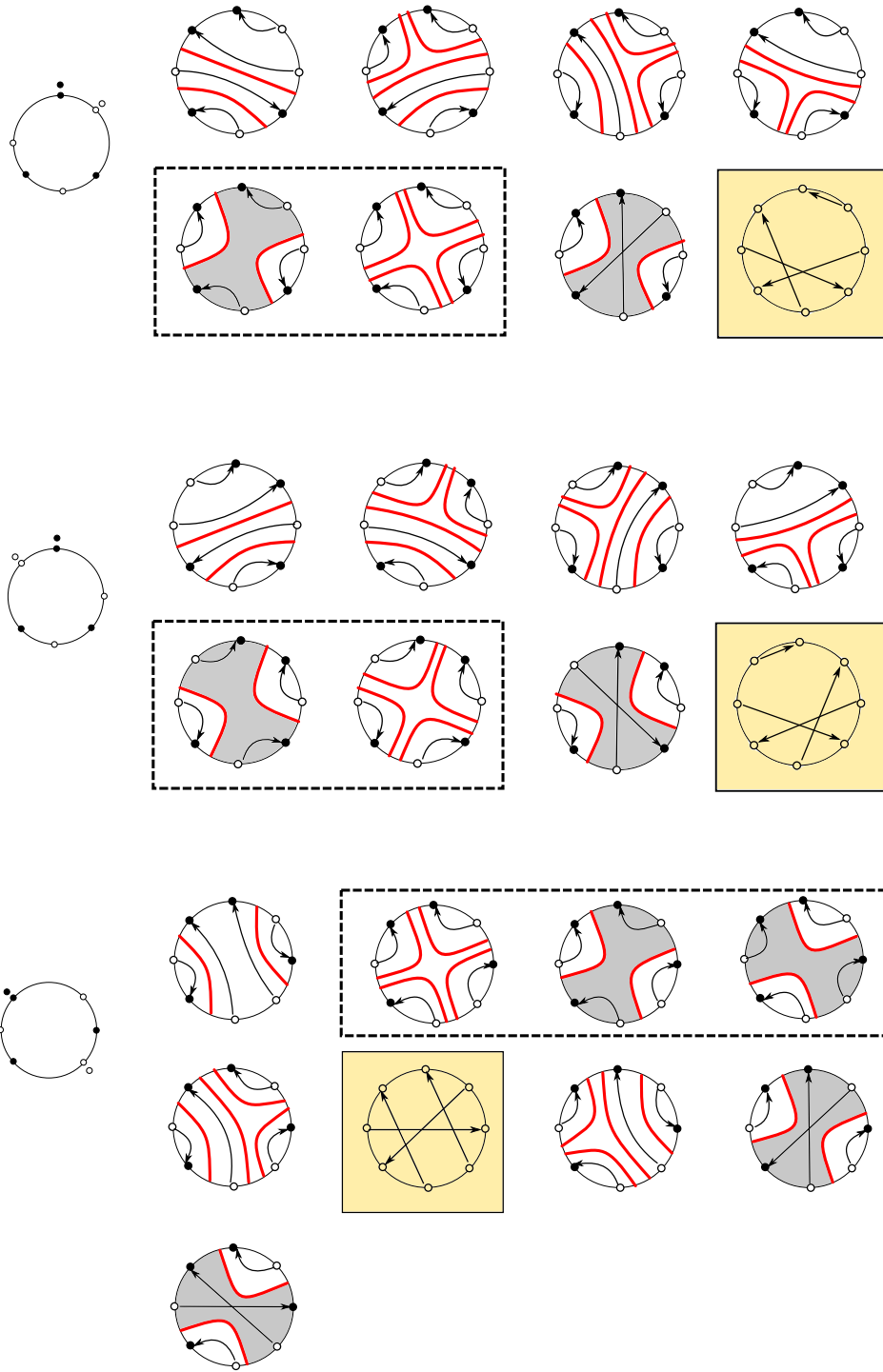


Figure 39: Diagram for $r = 4$ and $r' = 3$.

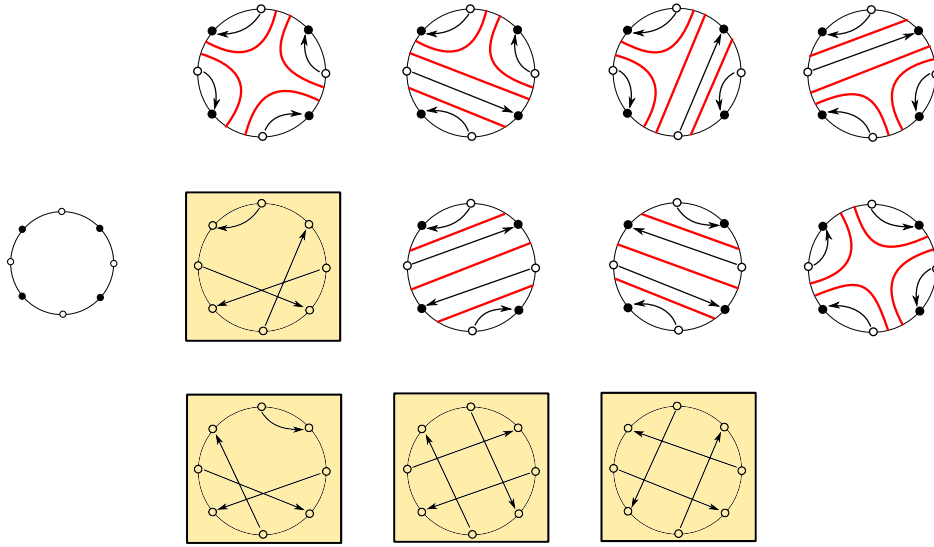


Figure 40: Diagram for $r = 4$ and $r' = 4$.

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